THE ELEMENTS OF ARITHMETIC.

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OF

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BY

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PREFACE.

This little book is intended to help Indian students in learning the principles and practice of Arithmetic. It contains all that is usually given in works on Arithmetic, together with such additional matter as has been deemed necessary to meet the wants of those for whom it is prepared. The explanations are given in a form such as would facilitate a clear and rational understanding of the principles; and the examples appended to each chapter would, it is hoped, furnish ample exercise for the student.

G. D. B.

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THE ELEMENTS OF ARITHMETIC.

INTRODUCTION.

ARTICLE 1. DEFINITION. ARITHMETIC is the science of *numbers*.

2. DEFS. A UNIT or UNITY means *one*.

A unit, or any part of unity, or any collection of units or parts of unity or both, is called a NUMBER.

Thus, *one, one-fourth, two, three-fourths, four and a half*, are numbers.

3. DEFS. A number consisting wholly of entire units is called an INTEGER or a WHOLE NUMBER.

A number not consisting wholly of entire units is called a FRACTION.

Thus, *one, two, three*, are integers; *half, three-fourths, two and a half*, are fractions.

4. DEFS. A number consisting of units or parts of units of no particular kind is called an ABSTRACT NUMBER.

A number consisting of units or parts of units of some particular kind is called a CONCRETE NUMBER.

Thus, *one, two, half*, considered simply, are abstract numbers; but in the instances, *one rupee, two yards, half a seer, one, two*, and *half*, are concrete numbers.

5. From Arts. 3 and 4 it will be seen, that every number must be either an integer or a fraction; and that each of these again must be either abstract or concrete. Thus, numbers may be divided into the following four classes:—

I. Abstract Integers.

II. Abstract Fractions.

III. Concrete Integers.

IV. Concrete Fractions.

6. DEF. ZERO, CIPHER, or NOUGHT, means *nothing* or no number.

It indicates the absence of number.

Being no number, it can neither be integral nor fractional, neither abstract nor concrete, in the ordinary sense of the terms. But in one sense, it may be said, in each case, to be of the same kind as the number whose absence it indicates.

Sometimes, for convenience of expression, zero is said to be a number. Thus we say that zero is the smallest number in Arithmetic. The meaning of such a proposition is this: zero being nothing, is less than any number that we can take, and is, therefore, the least of all numbers.

7. DEF. INFINITY means an infinitely large number, or a number larger than any number that we can take.

It may, therefore, for convenience of expression, be considered as the largest possible number.

8. Hence, *all numbers* in Arithmetic *range between zero and infinity*.

9. To facilitate operations, numbers have been expressed by certain figures or symbols; thus,

one, two, three, four, &c., are expressed by the figures,

1, 2, 3, 4, &c.

DEFS. The art of expressing by *figures* any number which is given in *words* is called NOTATION.

The art of expressing in *words* any number which is given in *figures* is called NUMERATION.

10. When there are any two given numbers, amongst other numbers which may be produced by combining them in different ways, we must be able to know what number results—

(1) when they are taken together;

(2) when one of them is taken from the other;

(3) when one of them is taken a number of times equal to the other; and

(4) when it is counted how often one of them is contained in the other.

The operations for finding the results in the above four cases are defined below.

11. Def. ADDITION is the method of finding what number, called the SUM, results from the taking together of two or more given numbers called the SUMMANDS.

The sign +, read PLUS, placed between two numbers, indicates that they are to be added together. Thus $1 + 2$ means that *one* is to be added to *two*.

12. Def. SUBTRACTION is the method of finding what number, called the DIFFERENCE or REMAINDER, results from the taking of a smaller number, called the SUBTRAHEND, from a greater, called the MINUEND.

The sign −, read MINUS, placed between two numbers, indicates that the latter is to be subtracted from the former. Thus $2 - 1$ means that *one* is to be subtracted from *two*.

13. Def. MULTIPLICATION is the method of finding what number, called the PRODUCT, results from the taking of one given number, called the MULTIPLICAND, another given number of times. This last mentioned given number is called the MULTIPLIER. The multiplicand and the multiplier are both sometimes called the FACTORS of the product.

The sign ×, read INTO, placed between two numbers, indicates that the former is to be multiplied by the latter. Thus 2×3 means that *two* is to be multiplied by *three*.

14. Def. DIVISION is the method of finding what number of times one number, called the DIVISOR, is contained in another, called the DIVIDEND. The first mentioned number is called the QUOTIENT.

The sign ÷, read BY or DIVIDED BY, placed between two numbers, indicates that the former is to be divided by the latter. Thus $4 \div 2$ means that *four* is to be divided by *two*. So also, one number written above another with a line between them, indicates that the former is to be divided by the latter. Thus $\frac{4}{2}$ means the same thing as $4 \div 2$.

15. The four operations defined above, Addition, Subtraction, Multiplication, and Division, form the basis of all other Arithmetical operations, as will be seen in the sequel. These four are

accordingly called the FUNDAMENTAL OPERATIONS in Arithmetic. They will be considered with reference to the four classes of numbers in four separate Chapters. The other operations will be defined in their proper places as we proceed.

16. DEFS. Two or more numbers expressed in figures and connected by one or more signs of operation, form what is called an EXPRESSION.

Thus $1 + 2$ is an expression.

When an expression is equal to another number or expression, the sign =, read EQUAL, is placed between them to indicate this equality, and the whole expression consisting of the two equal numbers with the sign of equality placed between them, is called an EQUATION.

Thus $1 + 2 = 3$ is an equation.

The signs $\because$ and $\therefore$ mean BECAUSE and THEREFORE respectively.

CHAPTER I.

THE FUNDAMENTAL OPERATIONS WITH ABSTRACT INTEGERS. MEASURES AND MULTIPLES.

SECTION I. NOTATION AND NUMERATION.

17. As every integer consists wholly of entire units, we see that *one* is the least integer ; the next integer *two* is formed by adding *one* to *one;* the next, *three*, by adding *one* to *two* ; and so on : and generally, *every succeeding integer is formed by adding unity to the preceding.*

Thus we get integers from *one* to *one hundred* and upwards without limit.

18. To express integers by symbols, the following are the three possible modes that we can adopt :—

1st. We can take a symbol for unity, and express every other number by repetition of this symbol.

2nd. We can take a separate symbol for every separate number.

3rd. We can take separate symbols for some of the integers, and express others by combinations of these.

The first and the second modes are obviously *complete*, that is, by them we can express *all* numbers ; but they are also obviously *inconvenient*, when we have to express large numbers ; for the first mode requires much space, and the second, the use of a large number of distinct symbols. Accordingly, these two modes have never been adopted, and we have only the third mode left for us.

But adopting the third mode, there still remains the question, —how many different symbols shall we take, and what mode of combining them shall we adopt.

The Greeks answered this question in one way, and invented their system of Notation. This is a complicated system, and so it has become obsolete.

The Romans answered it in another way, and invented their system of Notation. This too is not quite simple; but it is not so complicated as the Greek system, and is still retained in use occasionally, as in indicating the hour marks on dials of clocks and watches.

The Hindus answered the question in a third way, and invented their system of Notation. This system was borrowed from them by the Arabs, and from these last, by the nations of Europe, and is on that account sometimes called the Arabic system of Notation. It is now the *Common System of Notation* in almost all civilized countries. It is the best system of Notation that has been invented. We proceed to describe it in the next Article.

19. THE COMMON SYSTEM OF NOTATION.

In this system, the numbers *one, two, three, four, five, six, seven, eight, nine,* and *zero,* are expressed by the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, which are called DIGITS; and all other numbers are expressed by combinations of these figures according to the following Rule of *Convention*:—

RULE. A figure in the foremost place towards the right has its simple value, a figure removed *one* place towards the left has its value increased *tenfold,* a figure removed *two* places towards the left has its value increased ten times ten fold or a *hundredfold,* and so on; the value of a digit increasing *tenfold* at each step of removal towards the left: and zero may stand in any of these places except the last on the left, to show the absence of significant digits in those places, and to indicate the proper places for the digits to the left.

DEF. This increased value which a figure has in consequence of its *position,* is called its LOCAL VALUE, as distinguished from the value which it has when standing alone, and which is called its INTRINSIC VALUE.

Let us now see if this system is *complete* and *convenient.*

20. This system of Notation is *complete.* For we can express every integer in this system. Thus:—

One, two, three, four, five, six, seven, eight, nine, are expressed by 1, 2, 3, 4, 5, 6, 7, 8, 9. *Ten* is expressed by 10; for 0 standing in the foremost place on the

right, 1 is removed one place towards the left, and means ten times 1 or ten, and there is nothing more besides. *Eleven* is expressed by 11 ; for the 1 on the right means 1 unit, and the other 1 means ten times 1 or ten units, and the whole therefore means ten and one or eleven. Similarly, *twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen,* and *nineteen* are expressed by 12, 13, 14, 15, 16, 17, 18, and 19.

Twenty being two tens, will be expressed by 20.

So numbers from *twenty-one* to *ninety-nine* will be expressed by combinations of two digits, thus: 21, 22,......29,30,31... ....99.

Ninety-nine is the largest number that can be expressed by two digits. The next number *one hundred* will be expressed by 100 ; for the 1 standing to the left of two zeros has its value increased a hundredfold, and there is nothing more besides.

One hundred and one will be represented by 101; for the first 1 on the right is 1 unit, and the last 1 on the left, being removed two places towards the left, is 1 hundred, and there is nothing more. Similarly numbers from *one hundred and two* to *one hundred and nine* will be represented by 102, 103, 104, 105, 106, 107, 108, and 109.

One hundred and ten will be represented by 110 ; for the last 1 on the left is 1 hundred, the next is 1 ten, and there is nothing more. *One hundred and eleven* will be represented by 111, as the last 1 on the left means 1 hundred, the next to its right, 1 ten, and the next after it, 1 unit.

Similarly, numbers from *one hundred and twelve* to *nine hundred and ninety-nine* will be represented by combinations of three digits, thus: 112, 113, ..., 120, 121, ... 199, 200, 201, ... 999.

Nine hundred and ninety-nine is the largest number that can be expressed by three digits. The next number *one thousand* will be represented by 1000. And so on.

Thus all integers can be expressed in this system.

21. The Common System of Notation is also highly *convenient*, as will appear from the following considerations.

In ordinary language, the first nine numbers are named *one, two, &c., nine;* the next is named *ten;* the next nine numbers are named *eleven, twelve, thirteen, &c., nineteen, i. e.,* with slight modifications, *one and ten, two and ten, three and ten, &c., nine and ten.*

The next number is named *twenty* or *two tens,* and those after it are named *twenty-one, twenty-two, &c., thirty, thirty-one &c., ninety-nine, i. e., two tens and one, two tens and two, &c., three tens, three tens and one, &c., nine tens and nine.* The next number is *one hundred,* and those after it are named *one hundred and one, one hundred and two, &c., nine hundred and ninety-nine.* And so on.

Thus in common language, an integer is named by naming separately and explicitly the number of *units,* the number of *tens,* the number of *hundreds,* the number of *thousands,* &c., that it contains, in the order commencing with the highest, none of these numbers being greater than nine, and some of them being sometimes wanting.

And we have seen that in the Common System of Notation, an integer is expressed by expressing separately and explicitly the number of *units,* the number of *tens,* the number of *hundreds,* the number of *thousands,* &c., that it contains, in the order commencing with the highest, and proceeding from the left to the right, none of these numbers being greater than nine, and some of them being sometimes zeros.

Hence we see that the mode of *expressing* a number in the Common System of Notation is just the same as the mode of *naming* it in common language, the number ten forming the basis of both; and thus we can pass from the name to the symbolical expression, and *vice versa,* without any difficulty, taking care only, in Notation to fill up the vacant places with zeros, and in Numeration to observe the indications of zeros.

It is this easy convertibility of the names of numbers to their symbolical expressions, and *vice versa,* that makes this system of Notation so peculiarly convenient.

22. From Arts. 19–21 we see that when a number is expressed by figures, the place of the foremost figure on the right is the units' place, that of the second to the left is the tens' place, that of the third, the hundreds' place, and so on, as will be seen in the following Table, called the Numeration Table :—

&c.	Trillions.	Hundreds of Thousands of Billions.	Tens of Thousands of Billions.	Thousands of Billions.	Hundreds of Billions.	Tens of Billions.	Billions.	Hundreds of Thousands of Millions.	Tens of Thousands of Millions.	Thousands of Millions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
&c.	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
		Billions.						Millions.						Units.					

The period of 6 places from the 19th to the 24th consists of *trillions* ; the next period of 6 places consists of *quadrillions*. the next period, of *quintillions* ; &c.

Hence a number expressed in figures can be readily analyzed into its constituent units, tens, hundreds, &c., by separating its digits, and putting for each its proper local value.

Thus for example let it be required to analyze 302064 into its units, tens, hundreds, &c.

We have 302064 = 3 hundreds of thousands + no ten of thousands
+ 2 thousands + no hundred + 6 tens + 4 units
= 300000 + 2000 + 60 + 4.

23. The Indian Numeration Table in common use runs thus :—

&c.	Hundreds of Crores.	Tens of Crores.	Crores.	Tens of Lacs.	Lacs.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
&c.	10	9	8	7	6	5	4	3	2	1

24. The following are the Rules for Notation and Numeration :—

RULE I. In Notation, write down in their proper places digits corresponding to the numbers expressing the units, tens, hundreds, &c., in the given number, commencing with the highest place, and fill up the vacant places with ciphers.

RULE II. In Numeration, after dividing by commas the figures composing the given number into periods of 3 figures each, commencing with the foremost figure on the right, read out the periods with their proper denominations, commencing from the left.

Example 1. Write down in figures, two thousand and fifty millions, six hundred and thirteen thousand, five hundred and nine.

Here the first or the highest number *two* expressing thousands of millions, belongs to the 10th place; the 9th place is vacant, there being no hundreds of millions in the number; there being fifty millions, *i. e.*, *five* tens of millions, the *five* belongs to the 8th place; the 7th place is vacant there being no millions over and above the fifty millions; the next number *six* expressing hundreds of thousands, belongs to the 6th place; there being thirteen thousands, *i. e.*, *one* ten of thousands and *three* thousands, the *one* belongs to the 5th place and the *three* to the 4th; the 3rd place will be filled by the *five* of five hundred; the 2nd place is vacant; and the 1st place is filled by *nine*. The whole number will therefore be expressed thus,—2050613509.

Ex. 2. Write in words the meaning of 320506190.

Dividing the figures into periods we have 320, 506,190, and we see that the highest digit 3 belongs to the 9th or the hundreds of millions' place, the next, 2, to the tens of millions' place, &c.; the number will therefore be read thus :—Three hundred and twenty millions, five hundred and six thousand, one hundred and ninety.

25. The figure 0 annexed to the *right* of any integer expressed in figures, increases the value of every constituent digit tenfold by pushing it one place towards the left, and

thus increases the entire number tenfold. But annexed to the *left* of an integer, it has no effect.

Thus 150 is ten times 15.

26. We have seen that ten forms the basis of our ordinary mode of reckoning numbers. This is the case in almost every language, and was in all probability the fact which led to the invention of the Common System of Notation. But there arises the question, how to *account for* this fact. In answer to this, the following *theory* has been advanced, of which there is no reason to doubt the correctness.

In primitive society, in the infancy of Arithmetic, the *ten* fingers must have formed the readiest counters. But with the fingers, *one* man can count only up to *ten;* and to count more. he must have some means of indicating that his ten fingers have all been counted over. The readiest means available for such purpose would appear to have been the raising of the fingers of a *second* man, one each time that the first man counted over all his ten. In this way, eleven, twelve, &c., will be counted by *one* raised finger of the second man representing that all the ten fingers of the first have been once raised, and *one, two, &c.* raised fingers of the first. *Twenty* will be counted when the first man has counted over all his ten fingers again, and the second man has accordingly raised *two* fingers to indicate this. And so on. It was from this primitive mode of counting, that the almost universal practice of reckoning numbers by periods of ten seems to have originated.

This theory receives some confirmation also from the fact that the word *digit*, used to designate the elementary figures. also means a *finger*.

We may observe that the *digits*, including zero, in the successive places used to express a number in the Common System of Notation, correspond to the *numbers of the fingers* of the successive persons, raised to count the same number in the above mode. Thus, in 10,

the 0 in the 1st place corresponds to *no* raised finger of the 1st man
and 1...... 2nd........................ 1........................2nd ...

So, in 23,

the 3 in the 1st........................ 3......... fingers......1st ....
and 2...... 2nd........................ 2........................2nd ...

And so in other cases.

The inquiring student is referred for further information on the subject of Notation and Numeration to the Article on Arithmetic by Dr. Peacock in the Encyclopædia Metropolitana.

27. THE ROMAN SYSTEM OF NOTATION.

In this system, the following are the different symbols used, *viz.*,

I,	V,	X,	L,	C,	D or IↃ,	M or CIↃ,
for 1,	5,	10,	50,	100,	500,	1000,

respectively. Other numbers are expressed by combinations of these according to the following Rules :—

1st. A character followed by one or more characters of equal or less value indicates a number equal to the sum of the values of all these characters. A character preceded by another of less value denotes a number equal to the difference of their values.

Thus II $= 1 + 1 = 2$; IV $= 5 - 1 = 4$.

2nd. Every Ↄ annexed to IↃ increases the value of the latter tenfold. Every C prefixed and Ↄ annexed to CIↃ increases the value of the latter tenfold.

Thus IↃↃ = 5000, CCIↃↃ = 10,000.

3rd. A line $\overline{\quad}$ drawn over a character increases its value thousand-fold.

Thus $\overline{\text{I}} = 1000$.

The representation of one, two, three, by I, II, III, *i. e.*, by one, two, three strokes, is natural enough. The representation of ten, which forms a turning point in our computation, by X or a cross with one stroke crossing another, and of five, the half of ten, by the upper half of X, seems also to be equally natural.

To the beginner the following Table may be of use.

I,	II,	III,	IV,	V,	VI,	VII,	VIII,	IX,	X,
1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
XI,	XII,	XIII,	XIV,	XV,	XVI,	XVII,	XVIII,	XIX,	XX,
11,	12,	13,	14,	15,	16,	17,	18,	19,	20,

XXX,	XL,	L,	LX,	LXX,	LXXX,	XC,	C.
30,	40,	50,	60,	70,	80,	90,	100.

EXAMPLES I.

1. Express in figures the following :—

(1) Ten; twelve; fifteen; nineteen; twenty-eight; forty-four; fifty-six; sixty-one; eighty-four; ninety-two.

(2) One hundred and one; one hundred and ten; one hundred and fifty-four; three hundred; four hundred and five; five hundred and sixty; seven hundred and seventy-four.

(3) One thousand and one; two thousand and fifty-one; three thousand two hundred and sixty-three; four thousand; five thousand five hundred; six thousand seven hundred and eighty.

(4) One hundred thousand and one; two hundred thousand three hundred; three hundred and six thousand, seven hundred and nine; four hundred and fifty-six thousand and four; five hundred and sixty-seven thousand, four hundred and thirty-two.

(5) Two millions and one; three millions and twenty-nine; four millions, five hundred and sixty; five millions, six hundred thousand, and seventy-four; six millions, seven hundred and fifty-four thousand, three hundred and twenty-one.

(6) Three billions; four billions and five; five billions, six thousand, seven hundred and eight; seven billions, nine hundred and thirteen millions, five hundred and seventy-nine thousand, one hundred and thirty-five.

(7) Nineteen trillions; twenty trillions and twenty-four; thirty-one trillions, five hundred and fifty-six thousand seven hundred and nine millions, eight hundred and twenty-seven thousand five hundred and twenty.

(8) One lac and one; two lacs three thousand and three; five lacs sixty-one thousand seven hundred and twenty; fifteen lacs thirty thousand six hundred and twelve.

(9) Two crores and two; three crores five lacs seven thousand and nine; five crores sixty-four lacs thirty-two thousand one hundred and seventy-eight.

(10) Two hundred and sixteen crores fifty lacs sixteen thousand seven hundred and eighteen.

2. Express in words the following:—

(1) 18 ; 20 ; 37 ; 58 ; 69 ; 85 ; 97.
(2) 203 ; 340 ; 456 ; 690 ; 708 ; 991.
(3) 1009 ; 2029 ; 3690 ; 4862.
(4) 102030 ; 230450 ; 300004 ; 745621.
(5) 123456789 ; 987654321 ; 102030405.
(6) 2468101214 ; 248163264128.
(7) 50100200300400 ; 36912151821242730.
(8) 2305843008139952128 ; 137438691328.

3. Express in words according to the Indian Numeration Table the numbers in Examples (4) and (5).

4. Analyze by separating into units, tens, &c., the numbers in Examples (1), (2), and (3).

5. Express in Roman numerals the following:—

(1) 25 ; 33 ; 46 ; 87 ; 99.
(2) 101 ; 220 ; 314 ; 516 ; 999.
(3) 1001 ; 1856 ; 1864.

6. Express in figures the following :—

(1) XXVII ; XXXIV ; XXXXV ; XLVI.
(2) XCIX ; CCCI ; MXL ; DCL.
(3) MDCCCLVI ; CIↃDLXXXII ; MIX.

SECTION II. SIMPLE ADDITION.

28. Def. The Addition of abstract integers is called Simple Addition.

29. The following Table called the Addition Table should be committed to memory by the beginner.

Addition Table.

1 and 1 make 2	2 and 1 make 3	3 and 1 make 4	4 and 1 make 5	5 and 1 make 6	6 and 1 make 7	7 and 1 make 8	8 and 1 make 9	9 and 1 make 10
2 .. 3	2 .. 4	2 .. 5	2 .. 6	2 .. 7	2 .. 8	2 .. 9	2 .. 10	2 .. 11
3 .. 4	3 .. 5	3 .. 6	3 .. 7	3 .. 8	3 .. 9	3 .. 10	3 .. 11	3 .. 12
4 .. 5	4 .. 6	4 .. 7	4 .. 8	4 .. 9	4 .. 10	4 .. 11	4 .. 12	4 .. 13
5 .. 6	5 .. 7	5 .. 8	5 .. 9	5 .. 10	5 .. 11	5 .. 12	5 .. 13	5 .. 14
6 .. 7	6 .. 8	6 .. 9	6 .. 10	6 .. 11	6 .. 12	6 .. 13	6 .. 14	6 .. 15
7 .. 8	7 .. 9	7 .. 10	7 .. 11	7 .. 12	7 .. 13	7 .. 14	7 .. 15	7 .. 16
8 .. 9	8 .. 10	8 .. 11	8 .. 12	8 .. 13	8 .. 14	8 .. 15	8 .. 16	8 .. 17
9 .. 10	9 .. 11	9 .. 12	9 .. 13	9 .. 14	9 .. 15	9 .. 16	9 .. 17	9 .. 18

30. Rule for Addition. Write down in figures the several summands one under the other, so that units may be under units, tens under tens, &c., and draw a line below the last.

Add up the digits in the column of units, and place the figure in the units' place of the sum below the column of units; carry the number composed of the other digits, if any, add it along with the digits in the tens' column, and place the figure in the units' place of this sum below the tens' column; and carrying the number composed of the other digits, if any, to the next column, add it along with the digits in that column, and proceed as before. Repeat this process to the last column, and put the last sum in full. The entire number thus obtained is the sum required.

Ex. Add together 1099, 588, 689, and 2409.

By the Rule we have,

$$\begin{array}{r} 1099 \\ 588 \\ 689 \\ 2409 \\ \hline 4785 \end{array}$$

Reason for the Rule. To add numbers together is to add up successively the numbers of units, the numbers of tens, &c., that they contain; and this is done by adding together the digits in their units' places, those in their tens' places, &c., and placing the sums under those corresponding places, taking care, when any of these sums contains a number of a denomination higher than that of the digits added, to carry such number and add it to the sum of the digits of the next place. Thus in the Example above, the several summands contain $9 + 8 + 9 + 9$ units, $9 + 8 + 8 + 0$ tens, $0 + 5 + 6 + 4$ hundreds, and $1 + 2$ thousands, and the entire sum required will be the sum of these units, together with the sum of these tens, the sum of these hundreds, and the sum of these thousands. Now $9 + 8 + 9 + 9$ units $= 35$ units, *i. e.*, 5 units and 3 tens; 5 therefore is the digit that belongs to the units' place of the sum required, and we accordingly put 5 in the units' place. The 3 tens which the sum of the figures in the units' place gives, must now be added to the $9 + 8 + 8 + 0$ tens thus giving $3 + 9 + 8 + 8 + 0$ or 28 tens,

i. e., 8 tens and 20 tens or 2 hundreds; we accordingly put 8 in the tens' place, and carry 2 to be added to $0+5+6+4$ the sum of the digits in the hundreds' column. We thus get $2+0+5+6+4$ or 17 hundreds, *i. e.*, 7 hundreds and 10 hundreds or 1 thousand; 7 therefore is to be put in the hundreds' place, and 1 is to be carried and added to $1+2$ thousands. We thus get $1+1+2$ or 4 thousands, and 4 is accordingly placed in the thousands' place. The whole sum, therefore, is 4 thousands + 7 hundreds + 8 tens + 5 units, *i. e.*, 4785.

Worked out at full length, the process will stand thus:—

$$\begin{aligned}
1099 &= 1000 + 0 + 90 + 9 \\
588 &= 500 + 80 + 8 \\
689 &= 600 + 80 + 9 \\
2409 &= 2000 + 400 + 0 + 9 \\
\therefore \text{ the sum } &= 3000 + 1500 + 250 + 35 \\
&= 3000 + 1000 + 500 + 200 + 50 + 30 + 5 \\
&= 4000 + 700 + 80 + 5 \\
&= 4785
\end{aligned}$$

31. Proof. The correctness of the result in Addition may be tested thus:—

Add the several summands omitting one of them; to the sum thus obtained add the number omitted; and then if this last sum is the same as the sum of all the numbers, the result is in all probability correct.

The reason for this is obvious.

Ex. Add together 359, 1267, 486, 29.

Operation	Proof
359	359
1267	1267
486	486
29	29
2141	1782
	359
	2141

Ex. II.

1. Add

(1)	(2)	(3)	(4)	(5)
1	11	21	11	13
2	12	22	21	35
3	13	23	31	57
4	14	24	41	79
5	15	25	51	911
6	16	26	61	113
7	17	27	71	335
8	18	28	81	577
9	19	29	91	799

(6)	(7)	(8)	(9)
123	135	147	999
321	531	258	777
456	790	741	555
654	97	852	333
789	218	159	111
987	842	951	222

(10)	(11)	(12)	(13)
6789	1357	1234	15720
9876	7531	567	9134
6798	3175	89	720
6978	1537	1011	1516
9678	5173	121	9208
6879	7135	31	872

(14)	(15)	(16)
1000001	727421	12345
5200025	483853	6789
8630368	959496	1011
1974791	416151	12131
1250521	752726	4151
9600069	986383	61718
7000007	1303	404

(17)	(18)
123456789	121144
987654321	169196
432159876	225256
123456789	289324
341258967	361400
412359678	441484

2. Add together:—Six billions, five hundred and ninety-five millions, twenty-one thousand eight hundred and eighty-nine; two hundred and twenty thousand six hundred millions, one thousand and twenty-eight; fifty-six millions and fifty-six; one hundred and ninety-eight millions, one hundred and ninety-eight; sixty thousand and fifty.

3. Add together:—Two crores ten lacs fifty thousand and five; sixty-six lacs eleven thousand and seven; twenty-eight lacs seven thousand and five; fifty crores sixty lacs and seventy.

4. Add together:—123, 246, 4812, and 9624; also 456, 912, 1824, and 3648.

5. Find the sum of 65793, 752810, 446602, and 3979; also of 89210, 3579, 1012, and 201623.

6. Add together the sums of 35, 55, and 75; 44, 64, and 84; 12, 24, and 36; and 29, 39, and 49.

7. Add together the values of $1 + 2 + 3 + 4$, $11 + 12 + 13 + 14$, $21 + 22 + 23 + 24$, and $31 + 32 + 33 + 34$; also of $2 + 4 + 6$, $1 + 3 + 5$, $8 + 10 + 12$, $7 + 9 + 11$, and $2 + 4 + 8$.

8. Find the sum of 19 repeated 9 times; of 21 repeated 11 times; of 32 repeated 8 times; of 64 repeated 8 times; and of 40 repeated 9 times. Find also the sum of these sums.

SECTION III. SIMPLE SUBTRACTION.

32. Defs. The Subtraction of abstract integers or of concrete integers of the same denomination is called Simple Subtraction.

When a number is subtracted from another, then from the remainder so obtained, then from the next remainder, and so on, as long as the operation can be carried on, the entire process is called Continued Subtraction. It shews how often one number can be taken from another, or, in other words, how often it is contained in that other.

33. Def. The sign $\overline{\quad\quad}$ is called a Vinculum, and the signs ( ) and { } are called Brackets. When numbers are placed under the former, or are enclosed within the latter, it is to be understood that the operations indicated within the sign

are to be performed first, and the resulting number is then to be affected by the operations indicated outside.

Thus, $7 - \overline{5 - 2}$ or $7 - (5 - 2)$ means that 2 is to be first subtracted from 5, and then the result 3 is to be subtracted from 7.

34. PROPOSITION I. If the same number be added to, or subtracted from, both the minuend and the subtrahend, the difference remains unchanged.

Thus, $7 - 5 = 2$; and $7 + 2 - (5 + 2) = 9 - 7 = 2$ also;
and so also $7 - 3 - (5 - 3) = 4 - 2 = 2$.

PROP. II. Subtrahend + remainder = minuend.

PROP. III. Subtrahend = minuend − remainder.

For the minuend being made up of the subtrahend and the remainder, if the remainder is taken from it, there remains the subtrahend.

PROP. IV. If the same number be added to and subtracted from any number, the latter remains unchanged.

35. The following Table, called the Subtraction Table, should be committed to memory by the beginner.

SUBTRACTION TABLE.

1 from 1 leaves 0	2 from 2 leave 0	3 from 3 leave 0	4 from 4 leave 0	5 from 5 leave 0	6 from 6 leave 0	7 from 7 leave 0	8 from 8 leave 0	9 from 9 leave 0
2 .. 1	3 .. 1	4 .. 1	5 .. 1	6 .. 1	7 .. 1	8 .. 1	9 .. 1	10 .. 1
3 .. 2	4 .. 2	5 .. 2	6 .. 2	7 .. 2	8 ... 2	9 .. 2	10 .. 2	11 .. 2
4 . 3	5 .. 3	6 .. 3	7 .. 3	8 .. 3	9 .. 3	10 .. 3	11 .. 3	12 .. 3
5 . 4	6 .. 4	7 .. 4	8 .. 4	9 .. 4	10 .. 4	11 .. 4	12 .. 4	13 .. 4
6 .. 5	7 .. 5	8 .. 5	9 .. 5	10 .. 5	11 .. 5	12 .. 5	13 .. 5	14 .. 5
7 .. 6	8 .. 6	9 .. 6	10 .. 6	11 .. 6	12 .. 6	13 .. 6	14 .. 6	15 .. 6
8 .. 7	9 .. 7	10 .. 7	11 .. 7	12 .. 7	13 .. 7	14 .. 7	15 .. 7	16 .. 7
9 .. 8	10 .. 8	11 .. 8	12 .. 8	13 .. 8	14 .. 8	15 .. 8	16 .. 8	17 .. 8
0 .. 9	11 .. 9	12 .. 9	13 .. 9	14 .. 9	15 .. 9	16 .. 9	17 .. 9	18 .. 9

36. **RULE FOR SUBTRACTION. Place the smaller number under the greater, so that units may be under units, tens under tens, &c., and draw a line below.**

Subtract each figure of the subtrahend, commencing with the units' figure, from the corresponding figure of the minuend, and write the difference below it. When any digit in the subtrahend is greater than the corresponding digit in the minuend, add 10 to the latter and then subtract the

lower digit from the sum so obtained, and put down the difference below, taking care to carry 1 and add it to the next figure in the subtrahend before subtracting it from the corresponding figure in the minuend. The entire number thus obtained is the difference required.

Ex. 1. Subtract 938 from 6183.

By the Rule we have
$$\begin{array}{r} 6183 \\ 938 \\ \hline 5245 \end{array}$$

Reason for the Rule. To subtract one number from another is to subtract successively the number of units, the number of tens, &c., that the former contains, from the number of units, the number of tens, &c., respectively in the latter; and this is done by subtracting the digits in the successive places in the subtrahend from the corresponding digits of the minuend, and placing the differences under those corresponding places. If any digit in the subtrahend is greater than the corresponding digit in the minuend, to make the subtraction of that digit possible, we add to the digit in the minuend 10 of its own denomination, *i.e.*, 1 of the *next higher* denomination, and to keep the difference unchanged, we add 1 of this *next higher* denomination to the subtrahend, (Art. 34, Prop. I), *i. e.*, add 1 to the next digit in the subtrahend, before subtracting it from the digit above it. Thus in the Example above, 3 units being less than 8 units, we add 10 units, *i. e.*, 1 ten to 3 making it 13, and taking 8 units from 13 units, we have 5 units left; we therefore put 5 in the units' place of the difference required. And having added 1 ten to the minuend, to keep the difference unchanged, we must add 1 ten to the subtrahend; accordingly we carry 1 and add it to the 3 tens, thus getting 4 tens, which taken from 8 tens, leave 4 tens; we therefore put 4 in the tens' place. Again, 1, *i. e.*, 1 hundred being less than 9 or 9 hundreds, we add 10 hundreds or 1 thousand to it, thus getting 11 hundreds, and taking 9 hundreds from this, we have 2 hundreds for the difference; so 2 is put in the hundreds' place. And having added 1 thousand to the minuend, we must add it also to the subtrahend; and taking this 1 thousand from 6 thousands, we have 5 thousands

for the difference, and so we put 5 in the thousands' place. The whole difference, therefore, is 5 thousands + 2 hundreds + 4 tens + 5 units, *i. e.*, 5245.

Worked out at full length, the process will stand thus :—

$$
\begin{array}{rlllll}
\text{Diff. of} & 6183 & & & & \\
\text{and} & 938 & & & & \\
= \text{diff. of} & 6000 + 100 + 80 + 3 & & & & \\
\text{and} & \phantom{6000 + {}} 900 + 30 + 8 & & & & \\
= \text{diff. of} & 6000 + & (1000 + 100) + & 80 & + (10 + 3) & \\
\text{and} & 1000 + & 900 & + (10 + 30) + & 8 & \\
= \text{diff. of} & 6000 + & 1100 + & 80 + & 13 & \\
\text{and} & 1000 + & 900 + & 40 + & 8 & \\
= & 5000 + & 200 + & 40 + & 5 & \\
= & 5245 & & & &
\end{array}
$$

Ex. 2. Find by Continued Subtraction how often 5 can be taken from 17.

```
17
 5
—
12.................rem. after 5 is taken once;
 5
—
 7 .........................................twice;
 5
—
 2 .........................................thrice;
```

and 5 being greater than the last remainder 2, cannot be taken from it any more.

Thus 5 can be taken 3 times from 17, and there remains the number 2 besides.

Ex. 3. What number must be added to 12 to produce 19?

Since by Art. 34, Prop. II,

the diff. between two numbers + the smaller = the greater,
∴ the no. reqd. = the diff. between 19 and 12
= 7

Ex. 4. What number must be subtracted from 21 to give 15?

Since by Art. 34, Prop. III,
the subtrahend = the minuend − the remainder,
∴ the no. reqd. = 21 − 15
= 6.

37. **Proof.** To test the correctness of the result obtained by Subtraction, add the subtrahend and the difference together: and if the sum equals the minuend, the result may be presumed to be correct.

The reason for this is clear from Art. 34, Prop. II.

Ex. Subtract 325 from 1203.

Operation	Proof
1203	325
325	878
878	1203

Ex. III.

1. Subtract the smaller number from the greater in the following Examples :—

(1)	18	(2)	27	(3)	30	(4)	47	(5)	67
	12		18		20		36		49
(6)	123	(7)	678	(8)	101	(9)	151	(10)	100
	45		9		91		88		99
(11)	202122	(12)	484950	(13)	657687	(14)	192837		
	23240		51520		566778		28416		
(15)	918273	(16)	900009	(17)	80604020	(18)	987654321		
	827364		98765		9070503		123456789		
		(19)	20004	(20)	40009005				
			10005		30008007				

2. (1) By how much is the number five thousand greater than five hundred ?

(2) By how much is the number five lacs greater than five thousand ?

(3) By how much is the number six crores greater than three lacs ?

3. What number must be added

to	78	to make	100 ;
...	75	...	120 ;
...	64	...	128 ;
...	120	...	240 ;
...	400	...	529 ;
...	890	...	1000 ;
..	725	...	2009 ;
...	99999	...	111111 ;
...	88888	...	222222 ;
...	50005	...	100000 ?

4. Find the difference between the sum and the difference of a million and a billion, and of a crore and a lac.

5. Find by Continued Subtraction the number of times that

(1)	90	is contained in	900.
(2)	99	...	999.
(3)	123	...	1234.
(4)	556	...	2556.
(5)	2578	...	8752.
(6)	1234	...	2876.
(7)	576	...	4000.
(8)	677	...	5000.
(9)	881	...	6400.
(10)	500	...	2640.

6. Find the last remainder in each of the above Examples.

7. (1) What number together with the sum of 15 and 16 will be equal to the difference between 2 and 200 ?

(2) What number together with the difference between 5 and 15 will be equal to the sum of 6 and 16 ?

8. What number must be subtracted

from	100	to give	67 ;
...	210	...	78 ;
...	320	...	89 ;
...	1029	...	890 ;
...	542	...	120 ;
...	3589	...	1234 ?

SECTION IV. SIMPLE MULTIPLICATION.

38. In Multiplication, one of the given numbers, namely, the multiplier *must always be an abstract number*. For it indicates the *number of times* that the multiplicand is to be repeated; and there would be no meaning in saying that a number is repeated a concrete number of times, such as *five rupees* times, or the like, as that would be absurd.

The multiplicand may be abstract or concrete, and the product will be abstract or concrete accordingly.

39. Defs. When the multiplicand is an abstract number, or a concrete number of one denomination only, the Multiplication is called Simple Multiplication.

When the product of two numbers is multiplied by a third number, then the product so obtained, by a fourth, and so on, the process is called Continued Multiplication, and the last product, the Continued Product of all the numbers.

40. Multiplication is a short method of adding any number repeated as a summand any number of times.

Thus 7×4 means 7 taken 4 times, *i. e.*, $7 + 7 + 7 + 7$, and is equal to 28.

41. To multiply one abstract number by another is the same thing as to multiply the latter by the former.

Thus 3×4 is the same as 4×3.

$$\begin{array}{lrl}
 & \text{For } 3 \times 4 = & 3 + 3 + 3 + 3 \\
 & = & 1 + 1 + 1 \\
 & & +1 + 1 + 1 \\
 & & +1 + 1 + 1 \\
 & & +1 + 1 + 1 \\
\textit{i. e.} & = & 4 \text{ horizontal rows of 3 ones each,} \\
\text{or} & = & 3 \text{ vertical rows of 4 ones each,} \\
\textit{i. e.} & = & 4 \text{ ones repeated 3 times} \\
 & = & 4 \times 3.
\end{array}$$

42. Defs. When an integer is the product of two or more other integers, it is called a Composite Number, and those other integers are called its Factors. When an integer has no such factors, it is called a Prime Number.

Thus, 6,9,10 are composite numbers, being composed of the factors 2 and 3, 3 and 3, 2 and 5, respectively;

and 2, 5, 7 are prime numbers.

The continued product of any number repeated as a factor, is called a POWER of that number, being called the FIRST, SECOND, THIRD, &c. POWER, according as the factor enters once, twice, thrice, &c. The second and the third powers of a number are respectively called the SQUARE and the CUBE of that number.

The power of a number is denoted by that number having above it to its right a number indicating how often the former is repeated as a factor, and this last number is called the INDEX of the power.

Thus,

2	is called the 1st power of 2, and is denoted by	2^1 or 2;
2×2 or 4	..............2nd..............................	2^2;
$2 \times 2 \times 2$ or 8	..............3rd..............................	2^3.
&c. &c.	&c.	&c.

43. When the multiplier is the product of a series of factors, the product of the multiplicand and the multiplier is obtained by multiplying the former by the first factor of the latter, then the product so obtained by the next factor, and so on, to the last factor of the multiplier.

Thus
$$\begin{aligned} 7 \times 6 &= 7+7 + 7+7 + 7+7 \\ &= (7+7)+(7+7)+(7+7) \\ &= (7\times 2)+(7\times 2)+(7\times 2) \\ &= (7\times 2)\times 3, \text{ 2 and 3 being the factors of 6.} \end{aligned}$$

44. PROP. I. A number $\times$ zero or zero $\times$ a number = zero.

For a number taken *no* number of times, or *nothing* taken any number of times, will only give *nothing* as the product.

PROP. II. A number is multiplied by 10, 100, &c., by affixing one, two, &c. ciphers to its right.

The reason is clear from Art. 25.

PROP. III. The product of any number and the sum of any two others is equal to the sum of the products of that number and each of the others.

Thus $5 \times (3+4) = 5 \times 3 + 5 \times 4$.

For $5 \times (3+4)$ means 5 taken as often as there are units in 3 and 4 taken together, *i. e.*, as often as there are units in 3 and also as often as there are units in 4;

$$\therefore\ 5 \times (3+4) = (5+5+5) + (5+5+5+5)$$
$$= 5 \text{ taken } 3 \text{ times } + 5 \text{ taken } 4 \text{ times}$$
$$= 5 \times 3 + 5 \times 4$$

45. The following Table called the Multiplication Table should be committed to memory.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100
11	22	33	44	55	66	77	88	99	110
12	24	36	48	60	72	84	96	108	120
13	26	39	52	65	78	91	104	117	130
14	28	42	56	70	84	98	112	126	140
15	30	45	60	75	90	105	120	135	150
16	32	48	64	80	96	112	128	144	160
17	34	51	68	85	102	119	136	153	170
18	36	54	72	90	108	126	144	162	180
19	38	57	76	95	114	133	152	171	190
20	40	60	80	100	120	140	160	180	200

	11	12	13	14	15	16	17	18	19	20
11	121	132	143	154	165	176	187	198	209	220
12		144	156	168	180	192	204	216	228	240
13			169	182	195	208	221	234	247	260
14				196	210	224	238	252	266	280
15					225	240	255	270	285	300
16						256	272	288	304	320
17							289	306	323	340
18								324	342	360
19									361	380
20										400

Note.—The Multiplication Table given above is in the form in which it is taught to boys in Bengal in their vernacular.

46. Rule for Multiplication. Write down the multiplier under the multiplicand, so that units may be under units, tens under tens, &c., and draw a line below.

Multiply every figure of the multiplicand beginning with the units', by the units' figure of the multiplier ; place the units' figure of the first product under the figure of the multiplier that is being used ; carry its tens' figure, if any, add it to the next product, and place the units' figure of that product thus increased in the next place to the left ; and carrying the tens' figure, if any, add it to the next product, and proceed as before, down to the last product, and write that product, increased as above, in full.

Repeat the same process with the tens', hundreds', &c. figures of the multiplier, taking care to place the first digit in each case under the figure of the multiplier that is being used.

Add up the several lines of figures as in Addition, and the sum will be the product required.

Ex. 1. Multiply 5304 by 2039.

By the Rule we have,

```
    5304
    2039
   -----
   47736
  15912
 0000
10608
--------
10814856
```

Reason for the Rule. To multiply one number by another is to multiply the number of units, the number of tens, &c., contained in the former, first by the number of units, then by the number of tens, &c., in the latter, and then to add up these partial products; and this is done by multiplying every figure of the former by each figure of the latter, and adding up the several products, *regard being had to the local values of the figures.*

Thus in the Example above, the required product will be obtained by taking 5304 first 9 times, then 30 times, and then 2000 times, and adding up the partial products thus obtained.

Now to obtain 9 times 5304, we take 9 times 4 units, thus getting 36 units, *i. e.*, 3 tens and 6 units, and we ∴ put 6 in the units' place; we then take 9 times 0 tens, *i. e.*, 0 tens, which with the 3 tens in hand give 3 tens, and we ∴ put 3 in the tens' place; we next take 9 times 3 hundreds, thus getting 27 hundreds, *i. e.*, 2 thousands and 7 hundreds, and we ∴ put 7 in the hundreds' place; we lastly take 9 times 5 thousands, or 45 thousands, which with the 2 thousands in hand, give 47 thousands, and we ∴ put 7 and 4 in the thousands' and tens of thousands' places respectively. Thus we have 9 times 5304 = 47736.

Next, to obtain 30 times 5304, we take 3 times 5304 or 15912, which is obtained in the same way as 9 times 5304, and then multiply 15912 by 10, thus getting 159120 for the true partial product. (Art. 43.) The cipher at the end of this true partial product is omitted for convenience's sake, and the multiplication by 10 is indicated by removing the digits in the product of 5304 and 3 one step towards the left.

The next line consists of zeros, as it consists of the product of 5304 and 0 hundreds.

Lastly, to obtain 2000 times 5304, we take 2 times 5304 or 10608, and then multiply 10608 by 1000, thus getting 10608000 for the true partial product. We omit the 3 ciphers at the end for the sake of convenience, and indicate the multiplication by 1000 by removing the digits in the product of 5304 and 2 three places to the left.

Worked out at length, the process will stand thus :—

$$\begin{array}{r}
5304\\
2039\\
\hline
47736\\
159120\\
000000\\
10608000\\
\hline
10814856
\end{array}$$

Or more fully thus :—

$$\text{Since } 2039 = 2 \times 1000 + 0 \times 100 + 3 \times 10 + 9$$

$$\therefore 5304 \times 2039 = 5304 \times 9 + 5304 \times 3 \times 10 + 5304 \times 0 \times 100 + 5304 \times 2 \times 1000$$

$$\begin{aligned}
\text{Now } 5304 \times 9 &= 9 \times 5000 + 9 \times 300 + 9 \times 0 + 9 \times 4\\
&= 45000 + 2700 + 0 + 36\\
&= 47736
\end{aligned}$$

$$\begin{array}{rr}
\text{Similarly } 5304 \times 3 \times 10 = & 159120\\
5304 \times 0 \times 100 = & 000000\\
5304 \times 2 \times 1000 = & 10608000\\
\hline
\therefore \text{ the required prod.} = & 10814856
\end{array}$$

Ex. 2. Find the continued product of 21, 22, and 23.

$$\begin{array}{r}
21\\
22\\
\hline
42\\
42\\
\hline
462\\
23\\
\hline
1386\\
924\\
\hline
10626
\end{array}$$

47. ADDITIONAL RULES.

RULE. I. When a cipher occurs in the midst of other figures in the multiplier, the line of zeros in the series of partial products may be omitted.

RULE. II. When the multiplicand or multiplier or both contain ciphers at the end on the right, the product is obtained by first multiplying the numbers stripped of these ciphers, and then affixing to the right of the product thus obtained all the ciphers omitted.

Ex. Multiply 4300 by 2030

By the Rule we have

$$\begin{array}{r} 43 \\ 203 \\ \hline 129 \\ 86 \\ \hline 8729 \end{array}$$

8729; $\therefore$ the full product $= 8729000$

Reason for the Rule. By Art. 44, Prop. II,

$$4300 = 43 \times 100, \text{ and } 2030 = 203 \times 10;$$

$$\begin{aligned} \therefore 4300 \times 2030 &= 43 \times 100 \times 203 \times 10 = 43 \times 203 \times 100 \times 10 \\ &= 43 \times 203 \times 1000 \quad = 8729 \times 1000 \\ &= 8729000. \end{aligned}$$

48. MULTIPLICATION BY FACTORS.

RULE. When the multiplier is composed of simple factors, multiply the multiplicand by the first factor, then the first product by the second factor, and so on. The last product will be the one required.

The reason for this is clear from Art. 43.

Ex. Multiply 204 by 72.

Since $72 = 8 \times 9$, we have by the Rule

$$\begin{array}{r} 204 \\ 8 \\ \hline 1632 \\ 9 \\ \hline 14688 \end{array} \quad \ldots\ldots \text{ the prod. reqd.}$$

49. PROOF. The correctness of the result in Multiplication may be tested by the following method, called "CASTING OUT THE NINES":—

From the sum of the digits of the multiplicand subtract 9 as often as it can be subtracted, and set down the last remainder. Do the same thing with the multiplier and the product, and set down the corresponding remainders. Then if the last remainder left after successively subtracting 9 from the product of the remainders corresponding to the multiplier and the multiplicand, be the same as the remainder corresponding to the product, the operation has, very probably, been correctly performed.

Thus, referring to Ex. 1 in Art. 46,
we have the sum of the digits of the multiplicand = 12,
....multiplier = 14,
....product = 33,
and 12 - 9 = 3, the last rem. corresponding to the multiplicand.
14 - 9 = 5,.......................................multiplier.
and 33 - 9 = 24,
24 - 9 = 15,
15 - 9 = 6,.......................................product.
Now 3 × 5 = 15,
and 15 - 9 = 6 = ...

Hence we may say that the operation has been correctly performed.

The comparision of the remainders is usually exhibited thus :—

$$\begin{array}{ccc} & 6 & \\ 3 & \times & 5 \\ & 6 & \end{array}$$

The reason for this will be found in Art. 60.

Ex. IV.

1. Multiply—

(1) 123 by 4, 5, 6, 8, and 9.
(2) 456 by 7, 8. 9, 10 and 11.
(3) 789 by 3, 6, 9, 12, and 15.
(4) 123456789 by 5, 10, 15, 20, and 25.
(5) 987654321 by 8, 12, 16, 20, and 24.
(6) 123789 by 456, 654, and 465.

(7) 123456 by 789, 987, and 798.
(8) 1002003 by 405, 504, and 450.
(9) 4050607 by 809, 908, and 890.
(10) 80901001 by 1020, 3040, and 5060.
(11) 1200240048 by 12036 and 63021.
(12) 987654321 by 123456789 and 13579.
(13) 12340000 by 8900, 6700, and 4500.
(14) 5678000 by 91011000 and 121300.
(15) 89101100 by 50600 and 60500.

2. Find by Multiplication by factors the values of :—

(1) 123456×16. (2) 567890×32.
(3) 567890×64. (4) 987650×128.
(5) 69121518×81. (6) 13579×240.

3. Find the continued product of
(1) The first nine integers.
(2) 2, 4, 6, 8, 10, and 12.
(3) 3, 6, 9, 12, and 15.
(4) 1, 7, 11, 13, and 19.

4. Find the values of the following :—

(1) $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2$.
(2) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3$.
(3) $25^2 \times 15^3$.

5. Multiply one crore by one lac, and fifteen crores by sixteen lacs.

6. Multiply the sum of 1 lac and 1 crore by their difference, and also by twice that difference.

7. Multiply the difference between a lac and a thousand by that between a million and a crore.

SECTION V. SIMPLE DIVISION.

50. In Art. 14, the quotient has been defined to mean the number of times that the divisor is contained in the dividend. It may also mean the magnitude of each part of the dividend, when it is divided into the number of parts indicated by the divisor; and this, in fact, is its meaning, according to the sense in which Division is ordinarily understood. Thus, when

in common parlance we say, "Divide 8 by 2," the result we want is the *magnitude of each part* of 8 when it is divided into 2 equal parts, and not the *number of times* that 8 contains 2. The *numerical value* of the quotient, however, is the same, whichever meaning is attached to it.

To shew this, let us take as an example $13 \div 4$.

Then, by our Definition in Art. 14, $13 \div 4$ is 3 with 1 over, *i. e.*, 13 contains 4, 3 times, and there is 1 over; or in other words, 13 is 3 times 4, together with 1.

But 3 times 4 is the same as 4 times 3 (Art. 41).

Therefore, 13 is 4 times 3 together with 1; or in other words, 13 contains 3, 4 times, and there is 1 over; *i. e.*, 13 divided into 4 equal parts gives 3 as the magnitude of each part, and there is 1 remaining undivided.

Sometimes the quotient can have either meaning, and sometimes only one of the two, as will be seen in the next Article.

51. PROP. I. If the dividend be an abstract number, the divisor must also be an abstract number; and the quotient will be an abstract number, indicating either the number of times that the dividend contains the divisor, or the magnitude of each part of the dividend, when it is divided into the number of parts denoted by the divisor.

PROP. II. If the dividend be a concrete number, the divisor may be either abstract or concrete; and in the former case, the quotient will be a concrete number, indicating the magnitude of each part of the dividend when it is divided into the number of parts denoted by the divisor; and in the latter, it will be an abstract number denoting the number of times that the dividend contains the divisor.

The reason for this will be clear from the following considerations:—

Take as an example, the number *ten* divided by the number *two*.

In $10 \div 2$, the quotient 5 can have either meaning: it may mean, either the number of times that 10 contains 2, or the magnitude of each part of 10 when it is divided into 2 equal parts.

In $10 \div 2$ *rupees*, the operation would be unmeaning and absurd in either sense, as we cannot say that 2 *rupees* are contained in the abstract number 10 any number of times, nor can we divide the abstract number 10 into 2 *rupees* parts.

In 10 *rupees* $\div$ 2, the quotient 5 can only mean 5 *rupees*, *i. e.*, the magnitude of each part of the sum of 10 *rupees*, when it is divided into 2 equal parts. It cannot have the other meaning, for we cannot say that 10 *rupees* contain the abstract number 2 any number of times.

In 10 *rupees* $\div$ 2 *rupees*, the quotient 5 can only mean the abstract number 5, *i. e.*, the number of times that the sum of 2 *rupees* is contained in the amount 10 *rupees*. It cannot have the other meaning, for there can be no meaning in saying that 10 *rupees* are to be divided into 2 *rupees* parts.

52. DEFS. The Division of abstract integers or concrete integers of one denomination only is called SIMPLE DIVISION.

In the Division of integers, where the divisor is not contained in the dividend an integral number of times exactly, the number which remains after the former is taken from the latter the greatest possible number of times is called the REMAINDER.

Thus in $17 \div 5$, since 17 contains 5, 3 times and no more, with 2 over, the result of the operation is the quotient 3 with the remainder 2.

53. Division is a short method of subtracting one number repeatedly from another, to find how often the former is contained in the latter.

Thus, to divide 17 by 5 is to repeat the following operations of Subtraction,—(1) $17 - 5 = 12$; (2) $12 - 5 = 7$; (3) $7 - 5 = 2$; whence we see that 17 contains 5, 3 times with 2 over.

54. PROP. I. Dividend = divisor $\times$ quotient + remainder.

PROP. II. The product divided by the multiplicand or the multiplier = the multiplier or the multiplicand.

These evidently follow from the Definitions.

PROP. III. A number is divided by 10, 100, &c., by striking off one, two, &c. figures from its right, the figures struck off constituting the remainder in each case.

Thus take any number 1357.

Then 1357 $= 1350 + 7 = 135 \times 10 + 7$,

and also $= 1300 + 57 = 13 \times 100 + 57$,

.... = &c. ... &c.

Hence $1357 \div 10$ gives the quotient 135 with the remainder 7,

$1357 \div 100$.................... 13..........................57,

&c. ... &c. &c.

Prop. IV. The result of the division of the sum of any two numbers by a third is equal to the sum of the results of the division of those two numbers by the third number.

Thus taking any three numbers 8, 11, and 3,

$$\frac{8+11}{3} = \frac{8}{3} + \frac{11}{3}.$$

For, $\frac{8+11}{3}$

or $\frac{19}{3}$ gives the quotient 6, with 1 remaining to be divided by 3.

Now $\frac{8}{3}$.................... ..2......2..................................,

and $\frac{11}{3}$.... ....3......2..................................;

$\therefore \frac{8}{3} + \frac{11}{3}$... ....$2+3$......$2+2$...............................,

i. e.........................5......4..................................;

but $\frac{1}{3}$......................1......1..................................;

$\therefore \frac{8}{3} + \frac{11}{3}$..............$5+1$......1..................................,

i. e., the same result as $\frac{8+11}{3}$.

55. Rule for Division. Place the dividend and the divisor thus :—

Divisor) Dividend (

Mark off from the dividend on its left the least number of figures making a number not less than the divisor ; ascertain by trial the number of times that the divisor is contained in

it, by inquiring how often the figure in the highest place of the divisor is contained in the first figure, or, if that is too small, in the number composed of the first two figures on the left of the partial dividend ; place this number on the right of the dividend, as the first or the highest figure of the quotient ; and subtract the product of this number by the divisor from the portion of the dividend marked off.

To the right of the remainder thus obtained, annex the next figure of the dividend, and if the number thus formed be not less than the divisor, repeat with it the same process that was performed with the part of the dividend first marked off, putting the number now obtained for the quotient, as the next figure of the quotient required. If the above mentioned number be less than the divisor, annex a cipher to the quotient, bring down the next figure of the dividend, and so on, until the number becomes not less than the divisor.

Repeat this process until there is no more figure of the dividend to be annexed to the remainder. The last remainder, if any, will be the remainder after Division.

Ex. Divide 969708 by 482.

By the Rule we have

```
482)969708(2011
    964
    ---
      570
      482
      ---
       888
       482
       ---
       406
```

Reason for the Rule. To divide one number by another is to ascertain how many units of times not exceeding 9, how many tens of times not exceeding 9, how many hundreds of times not exceeding 9, &c., the divisor is contained in the dividend ; or in other words, to ascertain successively the digits in the units', tens', hundreds', &c., places of the quotient beginning with the highest.

Thus in the Example above, the portion of the dividend first marked off, 969, is really 969 *thousands*, and it contains 482,

2 *thousands* of times, with 5 *thousands* over ; accordingly we put 2 as the first figure or the figure in the *thousands'* place of the quotient.

Annexing now the next or the *hundreds'* figure, 7, of the dividend, to the difference between 969000 and 2000 times 482, *i. e.*, to 5000, we get 5700 or 57 *hundreds*, which contain 482 no *hundreds* of times ; we therefore put 0 as the *hundreds'* figure of the quotient.

Bringing down the next or the *tens'* figure 0, of the dividend, we get 5700 or 570 *tens*, which contain 482, 1 *ten* of times with 88 *tens* over ; we accordingly put 1 as the *tens'* figure of the quotient.

Lastly, annexing to the 88 tens or 880, the units figure 8 of the dividend, we obtain 888 *units*, which contain 482, 1 *unit* of time with 406 over ; and we accordingly put 1 in the *units'* place of the quotient.

Thus the complete quotient is 2011, and the final remainder, 406.

Worked out at length, the process will stand thus:—

$$
\begin{array}{l}
482)\ 969000 + 700 + 0 + 8\ (2000 + 0 + 10 + 1 \\
\underline{964000} \\
5000 + 700 = 5700 \\
\underline{4820} \\
880 + 8 = 888 \\
\underline{482} \\
406
\end{array}
$$

56. SHORT DIVISION. The mode of operation in Art. 55 is called LONG DIVISION. When the divisor is a small number not exceeding 20, the operation may be shortened into what is called SHORT DIVISION, by the following Rule :—

RULE. Place the divisor and the dividend thus :—

Divisor) Dividend.

Place the several figures of the quotient under the line drawn below the dividend, performing the several subtractions mentioned in the Rule in Art. 55 *mentally*.

Ex. Divide 205963 by 11.

By the Rule we have

$$11\,)\;\underline{205963}$$
$$18723\text{—}10 \text{ rem.}$$

57. **Division by Factors.**

Rule. When the divisor is composed of factors which are small numbers, divide the dividend by the first factor, then the first quotient by the second factor, then the second quotient by the third factor, and so on, to the last factor. The last quotient will be the one required; and the true remainder will be obtained by multiplying each remainder by all the divisors preceding its own, and adding together these products and the first remainder.

Ex. Divide 123248 by 72.

Since $72 = 6 \times 12 = 6 \times 4 \times 3$, we have by the Rule

$$72\begin{cases}3 \\ 4 \\ 6\end{cases}\begin{array}{|l}\underline{123248} \\ \underline{41082} - 2 \\ \underline{10270} - 2 \\ 1711 - 4\end{array}$$

The quotient is 1711 and the remainder is $4 \times 4 \times 3 + 2 \times 3 + 2 = 56$.

The *reason for the Rule* may be shewn thus :—

$$\begin{aligned}123248 &= 3 \times 41082 + 2 \text{ (Art. 54, Prop. I.)} \\ &= 3 \times (4 \times 10270 + 2) + 2 \\ &= 3 \times 4 \times 10270 + 3 \times 2 + 2 \text{ (Art. 44, Prop. III)} \\ &= 3 \times 4 \times (6 \times 1711 + 4) + 3 \times 2 + 2 \\ &= 3 \times 4 \times 6 \times 1711 + 3 \times 4 \times 4 + 3 \times 2 + 2 \\ &= 72 \times 1711 + 3 \times 4 \times 4 + 3 \times 2 + 2.\end{aligned}$$

58. **Additional Rules.**

Rule I. To divide a number by 10, 100, &c., cut off 1, 2, &c. figures from the right of the dividend, and the resulting number will be the quotient, and the number composed of the figures struck off, the remainder.

The reason for this is given in Art. 54, Prop. III.

Rule II. When the divisor ends in ciphers on its right, cut off these, and an equal number of figures from the right of the

dividend, and perform the division with these stripped off numbers. The quotient obtained will be the quotient required, and the true remainder will be obtained by annexing to the right of the remainder after division the figures of the dividend struck off.

Ex. Divide 126987 by 2300.

By the Rule we have

$$
\begin{array}{r}
23,00)\ 1269,87\ (55 \\
\underline{115\ } \\
119\ \\
\underline{115\ } \\
487
\end{array}
$$

The *reason for the Rule* may be shewn thus :—

$$
\begin{aligned}
126987 &= 126900 + 87, \\
&= 100 \times 1269 + 87\ ; \\
\text{and } 1269 &= 23 \times 55 + 4\ ; \\
\therefore 126987 &= 100 \times (23 \times 55 + 4) + 87 \\
&= 100 \times 23 \times 55 + 100 \times 4 + 87 \\
&= 2300 \times 55 + 400 + 87 \\
&= 2300 \times 55 + 487.
\end{aligned}
$$

59. Proof. The correctness of the result in Division may be tested thus :—

Multiply the divisor by the quotient, and add the remainder to the product. If the sum equals the dividend, the operation has been correctly performed.

The reason for this is clear from Art. 54, Prop. I.

The several partial products for this multiplictaion are already to be found in the operation of division, being the several subtrahends from below upward, in that operation.

Ex. Taking the Example in Art. 55, we have,

$$
\begin{array}{rl}
482 & \text{......divisor.} \\
\underline{2011} & \text{......quotient.} \\
482 & \\
482 & \\
\underline{9640} & \\
969302 & \\
\underline{406} & \text{......remainder.} \\
969708 & \text{......dividend.}
\end{array}
$$

60. We may here give the reason for the Rule of "Casting out the Nines," given in Art. 49, as a mode of testing the accuracy of results in Multiplication. It depends upon the following Proposition :—

Any number divided by 9 leaves the same remainder as the sum of its digits so divided leaves.

To prove this, take any number 28075.

Then,

$$
\begin{aligned}
28075 &= 20000 + 8000 + 70 + 5 \\
&= 2 \times 10000 + 8 \times 1000 + 7 \times 10 + 5 \\
&= 2 \times (9999 + 1) + 8 \times (999 + 1) + 7 \times (9 + 1) + 5 \\
&= 2 \times 9 \times 1111 + 2 + 8 \times 9 \times 111 + 8 + 7 \times 9 \times 1 + 7 + 5 \\
&= 9 \times (2 \times 1111 + 8 \times 111 + 7 \times 1) + 2 + 8 + 7 + 5;
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ 28075 \div 9 &= 9 \times (2 \times 1111 + 8 \times 111 + 7 \times 1) \div 9 \\
&\quad + (2 + 8 + 7 + 5) \div 9 \quad \text{(Art. 54, Prop. IV)} \\
&= 2 \times 1111 + 8 \times 111 + 7 \times 1 + (2 + 8 + 7 + 5) \div 9 .
\end{aligned}
$$

and $\therefore$ the remainder left after dividing 28075 by 9 is the same as the remainder left after dividing $2 + 8 + 7 + 5$ by 9.

Now, reverting to the Example in Art. 49, we have the multiplicand $5304 = 9 \times 589 + 3$,

where $3 =$ rem. left after dividing 5304 or $5 + 3 + 4$ by 9.

and the multiplier $2039 = 9 \times 226 + 5$,

where $5 =$ rem. left after dividing 2039 or $2 + 3 + 9$ by 9 :

$$
\begin{aligned}
\therefore \text{ the product} &= (9 \times 589 + 3) \times (9 \times 226 + 5) \\
&= 9 \times 226 \times (9 \times 589 + 3) + 5 \times (9 \times 589 + 3) \\
&= 9 \times 226 \times (9 \times 589 + 3) + 9 \times 589 \times 5 + 3 \times 5 ;
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ the product} \div 9 &= \frac{9 \times 226 \times (9 \times 589 + 3)}{9} + \frac{9 \times 589 \times 5}{9} + \frac{3 \times 5}{9} \\
&= 226 \times (9 \times 589 + 3) + 589 \times 5 + \frac{3 + 5}{9} ;
\end{aligned}
$$

i. e., if the product obtained is correct, the remainder left after dividing the product by 9, which is the same as the remainder left after dividing the sum of the digits in the product by 9, equals the remainder left after dividing 3×5 by 9.

If there is any error of 9, or if the digits in the product have their positions changed, this test will not enable us to detect the error in that case.

Ex. V.

1. Divide—

(1) 1234 by 2, 3, 4, and 5.
(2) 3456 by 3, 4, 5, and 6.
(3) 5678 by 5, 6, 7, and 8.
(4) 78910 by 7, 8, 9 and 10.
(5) 123456789 by 5, 10, 15, 20, and 25.
(6) 987654321 by 8, 12, 16, 20 and 24.
(7) 123789 by 456, 654, and 465.
(8) 123456 by 789, 987, and 798.
(9) 1002003 by 405, 504, and 450.
(10) 4050607 by 809, 908 and 890.
(11) 80901001 by 1020, 3040, and 5060.
(12) 1200240048 by 12036 and 63021.
(13) 987654321012345 by 123456 and 123456789.
(14) 12340000 by 8900, 6700, and 4500.
(15) 1000000000000 by 1111, 11110, and 111100.
(16) 111111111111 by 1111, 11110, 9999, and 99990.

2. Divide by the method of Short Division, by factors if necessary

(1) 3456 by 2, 3, 4, 5, and 6.
(2) 13579 by 4, 8, 12, and 16.
(3) 1000000 by 2, 3, 4, 5, 6, and 7.
(4) 111111111 by 8, 9, 10, 11, and 12.
(5) 2222222 by 13, 14, 15, 16, 17 and 18.
(6) 33333333 by 19, 20, 21, 22, and 24.
(7) 87654321910 by 32, 33, 34, and 35.
(8) 24681012 by 42, 44, 48, 49, and 51.

3. Divide one crore and one by one lac and one, and then the product of the quotient and the divisor by one thousand and one.

4. Divide the sum of a crore and a million by their difference.

5. Divide the product and the sum of a million and a lac by the difference between a million and a lac and the quotient of a million divided by a lac respectively.

SECTION VI. MEASURES AND MULTIPLES.

61. DEFS. A MEASURE or an ALIQUOT PART of an integer is an integer that is contained in the former an integral number of times exactly.

Hence unity is evidently a measure of every integer.

A COMMON MEASURE of two or more integers is an integer that is contained in each of them an integral number of times exactly.

The GREATEST COMMON MEASURE of two or more integers is the greatest integer that is contained in each of them an integral number of times exactly. The initials G. C. M. are often used for the words *greatest common measure.*

62. DEFS. A number exactly divisible by 2 is called an EVEN number.

A number not exactly divisible by 2 is called an ODD number.

Integers which have no common measure except unity are said to be PRIME to each other.

A number is said to be resolved into its ELEMENTARY FACTORS when it is resolved into factors which are prime numbers.

63. DEFS. A MULTIPLE of an integer is an integer that contains the former an integral number of times exactly.

Hence every integer is a multiple of unity.

A COMMON MULTIPLE of two or more integers is an integer that contains each of the former an integral number of times exactly.

The LEAST COMMON MULTIPLE of two or more integers is the least integer that contains each of the former an integral number of times exactly. The initials L. C. M. are often used for the words *least common multiple.*

64. PROP. I. A measure of a number is also a measure of any multiple of that number.

For, the measure is contained an integral number of times exactly in the number, and this again is contained an integral

number of times exactly in its multiple; hence the measure must be contained in the multiple an integral number of times exactly.

PROP. II. Every common measure of two numbers will measure their sum and their difference, and also any multiple of either.

Thus, 4 being a common measure of 20 and 28, (for $20 = 4 \times 5$, and $28 = 4 \times 7$),

$$28 + 20 = 7 \text{ times } 4 + 5 \text{ times } 4 = (7 + 5) \text{ times } 4$$
$$= 12 \text{ times } 4,$$

and $\therefore$ 4 measures $28 + 20$.

So, $28 - 20 = 7$ times $4 - 5$ times $4 = (7 - 5)$ times 4

$$= 2 \text{ times } 4,$$

and $\therefore$ 4 measure $28 - 20$.

And $\because$ 4 measure $28 + 20$ and $28 - 20$,

$\therefore$ by Prop. I, it will measure any multiple of either.

65. RULE FOR FINDING THE GREATEST COMMOM MEASURE OF TWO NUMBERS. Divide the greater number by the less, then divide the divisor by the remainder, if any, and so on, repeating this process till there is no remainder. The last divisor will be the greatest common measure required.

Ex. Find the G. C. M. of 98 and 70.

By the Rule we have

```
70) 98 (1
    70
    --
    28) 70 (2
        56
        --
        14) 28 (2
            28
```

$\therefore$ 14 is the G. C. M. required.

Reason for the Rule.

(1) The number 14 is a common measure of 98 and 70.

For 14 measures 28,
$\therefore$ 14 measures 2×28, and $\therefore$ $2 \times 28 + 14$ or 70 (Art. 64),
and $\therefore$ 14 measures $70 + 28$ or 98.

Hence 14 is a common measure of 98 and 70.

(2) It is also their greatest common measure.

For every number that measures 70 and 98,
measures $98 - 70$ or 28 (Art. 64),
and $\therefore$ measures 2×28 and $\therefore$ $70 - 2 \times 28$ or 14.

Now $\because$ no number greater than 14 can measure 14,
$\therefore$ 70 and 98 can have no common measure greater than 14 : and as 14 is itself a common measure of 70 and 98,
$\therefore$ it is their G. C. M.

66. Prop. I. Every common measure of two numbers measures their greatest common measure.

This is clear from the latter part of Art. 65.

Prop. II. Every measure of the greatest common measure of two numbers measures each of those numbers.

67. Rule for finding the Greatest Common Measure of Three or More Numbers. Find the greatest common measure of the first two numbers; then the greatest common measure of the greatest common measure so found and the third number; and so on. The last greatest common measure thus obtained will be the one required.

The *reason for the Rule* is clear.

Call the G. C. M. of the first two numbers the first G. C. M.

Then $\because$ by Art. 66, every common measure of the first G. C. M. and the 3rd number is a common measure of the three numbers,

$\therefore$ the greatest common measure of the first G. C. M. and the 3rd number is a common measure of the three numbers.

Again, $\because$ every common measure of the three numbers, being a common measure of the 1st and the 2nd, measures the G. C. M. of the 1st and the 2nd, and is therefore a common measure of the first G. C. M. and the 3rd number,

∴ every common measure of the three numbers is a measure of the greatest common measure of the first G. C. M. and the 3rd number.

Now, ∵ a number can have no measure greater than itself, ∴ no common measure of the three numbers can be greater than the greatest common measure of the first G. C. M. and the 3rd number.

And as the greatest common measure of the first G. C. M. and the 3rd number is itself a common measure of the three numbers,

∴ it is their G. C. M.

The same reasoning will hold good in the case of four or more numbers.

Ex. Find the G. C. M. of 32, 40, and 60.

By the Rule we have

```
32) 40 (1                8) 60 (7
    32                      56
    ——                      ——
     8) 32 (4                4) 8 (2
        32                      8
        ——                      —
```

∴ 4 is the G. C. M. required.

68. When any two numbers are divided by their greatest common measure, the quotients are prime to each other.

For, if these quotients have a common factor, the numbers themselves will have that common factor over and above their greatest common measure, which is contrary to the supposition of its being their greatest common measure, and is ∴ absurd.

69. **Rule for finding the Least Common Multiple of Two Numbers.** **Divide the product of the numbers by their greatest common measure, and the quotient will be the least common multiple required.**

Ex. Find the L. C. M. of 90 and 54.

By the Rule we have

54) 90 (1	54
54	90
36) 54 (1	18) 4860 (270
36	36
18) 36 (2	126
36	126

$\therefore$ 270 is the L. C. M. required.

Reason for the Rule. The least common multiple of any two given numbers being the least number that is divisible by each of them exactly, will contain as factors *all* the elementary factors of *each* of the given numbers *once*, and *only once*, and *no more* factors besides.

Now the product of any two given numbers being the continued product of *all* their factors, contains as factors all their elementary factors, every elementary factor *common to* these numbers being *twice* repeated, and *all other* elementary factors occurring *only once.* Hence if we divide the product of the numbers by the product of all these common elementary factors, *i. e.*, by the greatest common measure of the given numbers, we shall get their least common multiple.

Thus in the Example above,

$$90 = 5 \times 3 \times 3 \times 2, \qquad 54 = 3 \times 3 \times 3 \times 2\,;$$

$\therefore$ $90 \times 54 = 5 \times 3 \times 3 \times 2 \times 3 \times 3 \times 3 \times 2$, where $3 \times 3 \times 2$ or 18 occurs twice.

Now $\because$ the L. C. M. of 90 and 54 must contain the factors 5 and 3 and $3 \times 3 \times 2$ or 18 and nothing more besides,

$\therefore$ the L. C. M. required $= 90 \times 54 \div 18 = 270$.

70. Every common multiple of two numbers is a multiple of their least common multiple.

For, every common multiple of two numbers must contain as factors *all* the elementary factors of *each* of those numbers *at least once, i. e.*, must contain as factors *all* the factors of

their L. C. M. at least once, and it may contain any other factors besides;

and ∵ a number is divisible by the continued product of any number of its factors,

∴ every common multiple of two numbers is a multiple of their L. C. M.

71. Rule for finding the Least Common Multiple of Three or More Numbers. Find the least common multiple of the first two numbers; then the least common multiple of the least common multiple so found and the 3rd number: and so on. The last least common multiple thus obtained will be the one required.

The *reason for the Rule* is clear.

Call the L. C. M. of the first two numbers the first L. C. M.

Then ∵ every common multiple of the first L. C. M. and the 3rd number is a common multiple of the three numbers,

∴ the least common multiple of the first L. C. M. and the 3rd number is a common multiple of the three numbers.

Again, ∵ every common multiple of the three numbers, being a common multiple of the 1st and the 2nd, is a multiple of the L. C. M. of the 1st and the 2nd, and is therefore a common multiple of the first L. C. M. and the 3rd number.

∴ every common multiple of the three numbers is a multiple of the least common multiple of the first L. C. M. and the 3rd number.

Now ∵ a number can have no multiple smaller than itself,

∴ no common multiple of the three numbers can be less than the least common multiple of the first L. C. M. and the 3rd number.

And as the least common multiple of the first L. C. M. and the 3rd number is itself a common multiple of the three numbers,

∴ it is their L. C. M.

The same reasoning will hold good in the case of four or more numbers.

Ex. Find the L. C. M. of 32, 48, and 80.

By the Rule we have

```
32) 48 (1            48
    32               32
    --               --
    16) 32 (2        96
        32          144
        --      16)1536
                   ----
                     96   ∴ 96 is the L. C. M. of 32 and 48
80) 96 (1            96
    80               80
    --
    16) 80 (5   16)7680
        80          480   ∴ 480 is the L. C. M. reqd.
        --
```

72. Additional Rule. When the L. C. M. of several numbers is to be found, write down the numbers separated by commas in a line from left to right, leaving out such of them as are measures of any of the others.

Find by inspection the least integer greater than unity that measures any two or more of these remaining numbers, set it down on the left of the series, as a divisor; and put down in a line below each number the quotient after dividing it by the divisor, or the number itself when it is not exactly divisible by the divisor, thus obtaining a second series of numbers.

Treat this series exactly in the same way as the first; and proceed on, until a series is obtained in which the numbers are all prime to each other.

The continued product of all the divisors and all the numbers in the last series will be the L. C. M. required.

Ex. Find the L. C. M. of 9, 12, 15, 16, 20, 24, and 95.

By the Rule we have (leaving out 12 which measures 24)

2	9,	15,	16,	20,	24,	95
2	9,	15,	8,	10,	12,	95
2	9,	15,	4,	5,	6,	95
3	9,	15,	2,	5,	3,	95
5	3,	5,	2,	5,	1,	95
	3,	1,	2,	1,	1,	19

∴ the L. C. M. reqd. $= 2 \times 2 \times 2 \times 3 \times 5 \times 3 \times 1 \times 2 \times 1 \times 1 \times 19$
$= 13680.$

Reason for the Rule. The L. C. M. of the given numbers must contain as factors *all* the elementary factors of *each* of the given numbers *once* and *only once*, and *no more* factors besides. Hence we can leave out of consideration such of the given numbers as are measures of any of the others, for all the factors of these omitted numbers are contained in those that they measure. Now the continued product of the numbers that are retained, will contain as factors all their elementary factors, those that are prime to each other occurring only once, and those that are common to two or more numbers being repeated as often as there are numbers which have those common factors. To prevent this repetition of common factors, we divide those numbers that have any common factors by such factors; and, the continued product of these divisors or common factors each taken only once, and the ultimate quotients which are prime to each other, must be the L. C. M. required.

Thus in the Example above, 12 being a measure of 24 is left out.

The other numbers resolved into elementary factors stand thus :—

$$9 = 3 \times 3, \qquad 15 = 3 \times 5, \qquad 16 = 2 \times 2 \times 2 \times 2,$$
$$20 = 2 \times 2 \times 5, \quad 24 = 2 \times 2 \times 2 \times 3, \quad 95 = 5 \times 19.$$

Hence the required L. C. M. must contain as factors the two factors 3 and 3 of 9 ; the factor 5 only of 15, the other factor 3 having already been taken ; the four factors 2, 2, 2, and 2 of 16 ; no factor of 20, the factors 2, 2, and 5 having already been taken ; no factor of 24 for a similar reason ; and the factor 19 only of 95, the other factor 5 having already been taken ; and it must contain no other factors ; *i. e.*,

the L. C. M. reqd. $= 3 \times 3 \times 5 \times 2 \times 2 \times 2 \times 2 \times 19 = 13680$.

The object of the successive divisions by the common factors 2, 3, 5, is evidently to prevent the recurrence of those factors.

73. In the application of the Rule in Art. 72, the following Propositions will be of use.

Prop. I. A number is divisible by 2 if its units' figure is 0 or is divisible by 2.

For, taking any number 370, we have $370 = 10 \times 37 = 2 \times 5 \times 37$ which is evidently divisible by 2.

Again, taking any number 2589, we have

$$2589 = 2580 + 9 = 10 \times 258 + 9,$$

whereof the first part 10×258 containing 10 as a factor is evidently divisible by 2;

so that the whole number is divisible by 2 if its second part, *viz.*, the digit in its units' place, is divisible by 2.

Prop. II. A number is divisible by 3 or 9 if the sum of its digits is divisible by 3 or 9.

For, taking any number 2867, we have

$$\begin{aligned} 2867 &= 2 \times 1000 + 8 \times 100 + 6 \times 10 + 7 \\ &= 2 \times (999 + 1) + 8 \times (99 + 1) + 6 \times (9 + 1) + 7 \\ &= 2 \times 999 + 8 \times 99 + 6 \times 9 + (2 + 8 + 6 + 7) \\ &= 2 \times 9 \times 111 + 8 \times 9 \times 11 + 6 \times 9 \times 1 + (2 + 8 + 6 + 7) \\ &= 9 \times (2 \times 111 + 8 \times 11 + 6 \times 1) + (2 + 8 + 6 + 7). \end{aligned}$$

whereof the first part containing 9 as a factor is evidently divisible by 3 or 9;

so that the whole number is divisible by 3 or 9 if its second part, *viz.*, $2 + 8 + 6 + 7$, *i. e.*, the sum of its digits, is divisible by 3 or 9.

Prop. III. A number is divisible by 4 if its last two figures on the right are zeros or compose a number that is divisible by 4.

For, taking any number 28900, we have $28900 = 100 \times 289 = 4 \times 25 \times 289$ which is evidently divisible by 4.

Again, taking any number 78564, we have

$$78564 = 78500 + 64 = 100 \times 785 + 64,$$

whereof the first part is evidently divisible by 4;

so that the whole number is divisible by 4 if its second part, *i. e.*, the number composed of the digits in its tens' and units' places, is divisible by 4.

Prop. IV. A number is divisible by 5 if its units' figure is 0 or 5.

For, taking any number 3760, we have $3760 = 10 \times 376 = 2 \times 5 \times 376$ which is evidently divisible by 5.

Again, taking any number 4695, we have $4695 = 4690 + 5$, whereof the first part is evidently divisible by 5;

so that the whole number is divisible by 5 if its second part, *viz.*, its units' figure, is divisible by 5, *i. e.*, is 5 (that being the only digit that is so divisible).

PROP. V. A number is divisible by 6 if it is divisible by 2 and also by 3.

For $6 = 2 \times 3$, and 2 and 3 are prime to each other.

PROP. VI. A number is divisible by 8 if its last three figures on the right are zeros or compose a number that is divisible by 8.

For, taking any number 687000, we have $687000 = 1000 \times 687 = 8 \times 125 \times 687$ which is evidently divisible by 8.

Again, taking any number 987682, we have

$$987682 = 987000 + 682,$$

whereof the first part 987000 is divisible by 8, from what has been just shewn;

so that the whole number is divisible by 8 if its second part, *i. e.*, the number composed of the last three digits on its right, is divisible by 8.

PROP. VII. A number is divisible by 11 if the sums of the figures in its odd and even places differ by 0 or a multiple of 11.

To prove this Proposition, it is necessary first of all to establish the following truth:—

An odd power of ten + unity or an even power of ten − unity is divisible by eleven.

Thus,

$$10^1 + 1 = 10 + 1 = 11,$$
$$10^2 - 1 = 100 - 1 = 99 = 11 \times 9,$$
$$10^3 + 1 = 1000 + 1 = 999 + 1 + 1 = 990 + 9 + 2 = 11 \times 90 + 11$$
$$= 11 \times (90 + 1),$$
$$10^4 - 1 = 10000 - 1 = 9999 = 11 \times 909,$$
$$10^5 + 1 = 100000 + 1 = 99999 + 1 + 1 = 99990 + 9 + 2$$
$$= 11 \times 9090 + 11 = 11 \times (9090 + 1),$$

&c. &c.;

which are all evidently divisible by 11.

Now taking any number 259768, we have

$$\begin{aligned}
259768 &= 200000 + 50000 + 9000 + 700 + 60 + 8 \\
&= 2 \times 100000 + 5 \times 10000 + 9 \times 1000 + 7 \times 100 + 6 \times 10 + 8 \\
&= 2 \times 10^5 + 5 \times 10^4 + 9 \times 10^3 + 7 \times 10^2 + 6 \times 10^1 + 8 \\
&= 2 \times 10^5 + 2 - 2 + 5 \times 10^4 - 5 + 5 + 9 \times 10^3 + 9 - 9 \\
&\quad + 7 \times 10^2 - 7 + 7 + 6 \times 10^1 + 6 - 6 + 8 \text{ (Art. 34, Prop. IV)} \\
&= 2 \times (10^5 + 1) + 5 \times (10^4 - 1) + 9 \times (10^3 + 1) \\
&\quad + 7 \times (10^2 - 1) + 6 \times (10 + 1) + 8 + 7 + 5 - 6 - 9 - 2,
\end{aligned}$$

whereof the numbers within brackets are divisible by 11, as has been already shewn;

so that the whole number is divisible by 11 if $8 + 7 + 5 - 6 - 9 - 2$ is 0 or is divisible by 11, *i. e.*, if the sum of the figures in the odd places − the sum of the figures in the even places = 0 or a multiple of 11.

PROP. VIII. A number is divisible by the product of any factors prime to each other if it is divisible by each of them.

Thus 12 being equal to 3×4, and 3 being prime to 4, a number is divisible by 12 if it is divisible by 3 and 4.

But if the factors are not prime to each other, a number may be divisible by each of them, and yet it may not be divisible by their product.

Thus, taking two numbers 4 and 6 which are not prime to each other but have the factor 2 common to both, we see that the number 12 is divisible by 4 and also by 6, but is not divisible by 4×6. And the reason for this is obvious.

For, 12 being $2 \times 2 \times 3$, contains each of the factors of 4 and 6 which are prime to each other, namely 2 and 3, once, and besides these, it contains the other factor 2 which is common to both 4 and 6 *only once*, so that though 12 is divisible by each of the two 4 and 6, *when taken alone*, it is not divisible by their *product* which is 4×6 or $2 \times 2 \times 2 \times 3$, that is, which contains their common factor 2 *twice*, over and above their factors 2 and 3 which are prime to each other.

74. To ascertain whether any given number is a prime number :—

RULE. Write in a series all the integers from 1 to the given number.

After 2, go on marking every 2nd number with a dot; then every dotted number is evidently divisible by 2, and the rest are not.

Next take the next unmarked number 3, and go on marking every third number after it; then those that are marked this time are divisible by 3, and the rest are not.

Repeat the same process with every succeeding unmarked number down to the middle number, and if the given number still remains unmarked, it is a prime number.

The *reason for the Rule* is clear. For if the given number had any factor, it must have been marked when the dotting commenced from that factor.

This method also gives all the prime numbers less than the given number, and shews what numbers are the factors of those that are not prime.

This simple method we owe to Eratosthenes, a Greek mathematician, who called it his *sieve* for sifting out prime numbers. The advanced student will see that the method given above can be simplified still more.

Before applying this method which is tedious, Propositions I to VII of Art. 73 should be applied to every case.

Ex. Ascertain if 23 is a prime number.

By the Rule we have

$$1,\ 2,\ 3,\ \dot{4},\ 5,\ \ddot{6},\ 7,\ \dot{8},\ \dot{9},\ \ddot{10},\ 11,\ \ddot{12},\ 13,$$

$$\ddot{14},\ \dot{15},\ \dot{16},\ 17,\ \ddot{18},\ 19,\ \ddot{20},\ \ddot{21},\ \ddot{22},\ 23.$$

Thus we see that 23 is a prime number.

The student should notice that the following are all the prime numbers below 100:—

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43,
47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

75. With the help of Arts. 73 and 74 and the Multiplication Table, we can resolve numbers into their elementary factors.

Ex. . Resolve 234 into its elementary factors.

By Art. 73, 2 and 9 are factors of 234,

and $234 = 2 \times 117 = 2 \times 9 \times 13 = 2 \times 3 \times 3 \times 13$,

and these factors being all prime numbers, are elementary factors.

Ex. VI.

1. Find the G. C. M. of

(1) 24 and 30; 32 and 48; 24 and 60; 32 and 80; 28 and 70; 38 and 95; 95 and 133; 115 and 161.

(2) 84 and 144; 133 and 228; 175 and 300; 144 and 208; 171 and 247; 153 and 289; 171 and 380; 323 and 361

(3) 105 and 600; 56 and 128; 64 and 276; 121 and 209; 289 and 425; 289 and 867; 121 and 1331; 361 and 1805.

(4) 456 and 646; 442 and 646; 532 and 588; 516 and 817

(5) 1164 and 1455; 948 and 1659; 876 and 2190; 852 and 2343.

(6) 1414 and 2222; 1605 and 2247; 1359 and 3624; 2172 and 2896.

(7) 2796 and 2868; 2358 and 3006; 1245 and 2265; 2589 and 9493.

(8) 916 and 2164; 2164 and 6332; 9498 and 21426.

(9) 123456789 and 987654321; 999999 and 9903300.

2. Find the G. C. M. of

(1) 24, 30, and 48.
(2) 108, 144, and 192.
(3) 150, 225, and 365.
(4) 144, 180, 216, and 324.
(5) 171, 228, 342, and 380
(6) 198, 264, and 924.

3. Find the L. C. M. of

(1) 12 and 27; 14 and 42; 14 and 63; 18 and 42 18 and 78; 23 and 95; 35 and 119; 49 and 133.

(2) 78 and 117; 78 and 143; 75 and 225; 119 and 289; 361 and 570; 276 and 368.

(3) 852 and 2343; 948 and 1659; 1414 and 2222.

(4) 285714 and 999999; 10353 and 14875.

(5) 123456789 and 987654321.

4. Find the L. C. M. of

(1) 1, 2, 3, 4, 5, 6, 7, 8, and 9.
(2) 1, 3, 5, 7, 9, 11, 13, and 15.
(3) 2, 4, 6, 8, 10, 12, 14, and 16.
(4) 1, 4, 9, 16, 25, 36, 49, 64, and 81.
(5) 5, 10, 15, 20, 25, 30, 35, and 40.
(6) 3, 4, 8, 6, 9, 18, and 30.
(7) 8, 20, 176, 165, 45, and 1233.
(8) 8, 1328, 166, 121, 22, and 40.
(9) 9, 16, 23, 30, 37, 44, and 51.
(10) 10, 18, 26, 34, 42, and 51.
(11) 9, 24, 75, 144, 180, and 508.
(12) 8, 18, 22, 176, 540, and 550.

5. Resolve into elementary factors

(1) 92, 108, 111, 119, 171, and 189.
(2) 204, 216, 260, 289, and 304.
(3) 324, 361, 402, 403, and 404.
(4) 406, 407, 415, 512, and 618.
(5) 759, 781, 790, 828, and 945.

MISCELLANEOUS QUESTIONS AND EXAMPLES.

76. We have noticed in Art. 33 that when an expression contains brackets, it is to be understood that in simplifying it, we are to perform first of all the operations indicated within the brackets, and then the operations indicated outside. It now remains to notice a Rule of *convention* which relates to the order in which the operations in either case are to be performed. It is this :—

RULE. Within the brackets, perform first of all the operations of Division with the numbers immediately on the two sides of the sign $\div$; then perform all the operations of Multiplication with the resulting numbers on both sides of the sign $\times$; then add together all the resulting numbers that are preceded by the sign $-$, *i. e.*, are meant to be subtracted; and then subtract the sum so obtained from the sum of the other resulting numbers.

Having thus cleared the brackets, observe the same rule outside the brackets.

Ex. Simplify

$654 \div 3 \times 2 - 18 \div (2 \times 3) \times 4 + (51 \div 3 - 15) \times 8 - (12 \times 4 - 80 \div 2)$

By the Rule, the expression

$$\begin{aligned}
&= 654 \div 3 \times 2 - 18 \div 6 \times 4 + (17 - 15) \times 8 - (48 - 40) \\
&= 654 \div 3 \times 2 - 18 \div 6 \times 4 + 2 \times 8 - 8 \\
&= 218 \times 2 - 3 \times 4 + 2 \times 8 - 8 \\
&= 436 - 12 + 16 - 8 \\
&= 452 - 20 \\
&= 432.
\end{aligned}$$

77. In working out Examples given in language *other than the pure language of Arithmetic,* we must first of all clearly understand the question, and then *translate* it into the language of Arithmetic, and state clearly, concisely, and *methodically* all the steps of the process. The statement of the steps of the process must consist of connected and complete intelligible *sentences*, differing from ordinary sentences only in this respect, that signs, symbols, and abbreviations are for the most part used here instead of words. These remarks will be illustrated by the following Examples.

Ex. 1. A person A has in his box 300 rupees in silver and 200 pieces of currency notes each worth 50 rupees. Of this, the sum of 240 rupees belongs to another person B. C and D each owe 100 rupees to A, and A owes rupees 150 and 250 to E and F respectively. If A has no other property and owes nothing more to any body, how much is he really worth in rupees?

Here, evidently

$$\begin{aligned}
\text{the money } A \text{ is worth} &= \text{the amount in his possession} \\
&\quad + \text{the amounts due to him} \\
&\quad - \text{the amount not his own} \\
&\quad - \text{also the amounts he owes to others} \\
&= 300 \text{ rupees} + 200 \times 50 \text{ rupees} \\
&\quad + 100 \text{ rupees} + 100 \text{ rupees} \\
&\quad - 240 \text{ rupees} \\
&\quad - (150 \text{ rupees} + 250 \text{ rupees}) \\
&= (300 + 10000 + 200 - 240 - 400) \text{ rupees} \\
&= (10500 - 640) \text{ rupees} \\
&= 9860 \text{ rupees.}
\end{aligned}$$

Ex. 2. The sum of the ages of 3 boys A, B, and C is 30 years; the sum of the ages of A and B is 18; and that of the ages of A and C is 20. What is the age of each?

$$\begin{aligned}
&\text{Since age of } A + \text{age of } B + \text{age of } C = 30,\\
&\text{and } \ldots\ \ldots\ A + \ldots\ldots.. B \qquad = 18,\\
&\therefore \text{ by Subtraction} \qquad \text{age of } C = 12.
\end{aligned}$$

$$\begin{aligned}
&\text{Again } \because \text{ age of } A + \text{ age of } C = 20,\\
&\text{and} \qquad \text{age of } C = 12,\\
&\therefore \quad \text{age of } A \qquad = 8;\\
&\text{and } \because \text{ age of } A + \text{ age of } B = 18,\\
&\text{and age of } A \qquad = 8,\\
&\therefore \qquad \text{age of } B = 10.
\end{aligned}$$

Hence the ages of A, B, and C are 8, 10, and 12 years respectively.

Ex. 3. In a field there are 20 rows of trees, each row containing 25 trees. What is the total number of trees?

$$\begin{aligned}
\text{Here, the total no. reqd.} &= \text{the no. of trees in a row repeated as often as there are rows}\\
&= 25 \text{ repeated } 20 \text{ times}\\
&= 25 \times 20\\
&= 500.
\end{aligned}$$

Ex. 4. A bookseller having sold several copies of a book at 3 rupees a copy, finds that he has realized 63 rupees by the sale. How many copies did he sell?

$$\begin{aligned}
\text{Here the no. of copies reqd.} &= \text{the no. of times that 3 must be repeated to produce 63}\\
&= \text{the no. of times that 3 is contained in 63}\\
&= 63 \div 3\\
&= 21.
\end{aligned}$$

Ex. 5. A wheel 3 cubits in circumference is made to roll over a length of 18 cubits. How many revolutions does it make? And what length will it roll over in making 8 revolutions?

Here we must first of all know what length the wheel rolls over in one revolution.

Now one revolution is completed when the wheel, after having any particular point in its circumference in contact with the surface rolled over, comes to have the same point again in contact with that surface; and this happens after successive points on the whole circumference have come in contact, one after another, with points on the path rolled over.

Hence the length rolled over in one revolution = circumference of the wheel
= 3 cubits.

Therefore the no. of revolutions reqd. = the no. of times that 3 is contained in 18
$= 18 \div 3$
$= 6.$

And the length rolled over in 8 revolutions
$= 8 \times 3$ cubits
$= 24$ cubits.

Ex. 6. A man brought to the market a certain number of mangoes for sale. He sold 5 mangoes to A; to B, 3 more than to A; and to C, 2 less than to B; and found that he had sold half of the whole number. How many mangoes did he bring for sale?

Here,
the no. of mangoes sold to $A = 5$,
..............................$B = 5 + 3 = 8$,
and ....$C = 8 - 2 = 6$;
∴..............................in all $= 5 + 8 + 6$
$= 19$,
which = half the no. brought;
∴..................brought for sale $= 2 \times 19$
$= 38.$

Ex. 7. The first book of a poem contains 495 lines, and the second book, 900 lines. Find the largest number of lines that a page can contain, so that every page may contain the same number of lines, and each of the two books may consist of an integral number of pages.

Since each book is to consist of an integral number of pages, and every page is to contain the same number of lines,

the number of lines in a page must be a common measure of 495 and 900, and $\therefore$ the largest number of lines that a page can contain = the G. C. M. of 495 and 900

= 45, by the ordinary process for finding the G. C. M.

Ex. 8. Find the least number of rupees which can be divided equally among 2, 3, 4, 5, or 6 men.

Since the number required is to be divisible by 2, 3, 4, 5, and 6, it must be a common multiple of these numbers;

and $\because$ it is to be the least number that is so divisible,

$\therefore$ it must be their L. C. M. which by the ordinary Rule $= 4 \times 5 \times 3 = 60$.

Ex. VII.

I.

1. What is Arithmetic? Define the terms Unity and Number, and distinguish between Abstract and Concrete Numbers, and between Integers and Fractions.

2. Define Notation and Numeration, and describe briefly the Common System of Notation.

3. Write down in figures, twenty millions thirty thousand and forty, and read the result according to the Indian Numeration Table.

4. What is meant by the Local Value of a digit? How is this value affected by affixing ciphers to the right of a number?

5. Three boys *A*, *B*, and *C*, have each a certain number of marbles. The total number of marbles is 18, and of these *A* and *B* together have 9, and *A* and *C* together 12. How many marbles has each?

6. A gentleman whose age is 33 has two sons. The age of the second son is 6 years, and the difference between the age of the father and the sum of the ages of the sons is 18 years. Find the age of the first son.

7. A woman brought a certain number of mangoes to the market for sale. She sold 10 mangoes to Ram; to Syam 5 more than to Ram; and to Hari as many as to Ram and Syam together; and she found that had she sold 5 more to each of

the three, Ram, Syam, and Hari, there would have remained unsold just as many as she would have sold in that case. How many mangoes did she bring?

8. The number of boys in the first two classes of a school is 47. In the third class there are 15 boys less than in the first two classes taken together, and in the second class, there are 3 boys more than in the third. How many boys are there in each class?

II.

1. Define the terms Summand and Sum.

Add together :—Seventeen thousand three hundred and four; nineteen hundred and twenty; and eleven hundred and twelve.

2. In the Subtraction of integers, when any figure in the subtrahend is greater than the corresponding figure in the minuend, how do you proceed, and why in that way?

3. Shew that the sum of any two numbers added to their difference is equal to twice the greater number.

4. How do you account for the fact that the number ten forms the basis of our numerical computation?

5. The total number of pages in three books taken together is 129. The number of pages in the first two taken together is 62; and the third has 31 pages more than the second. How many pages are there in each?

6. A person had 25 rupees in his pocket. After the purchase of some books and stationery, he finds that he has only 5 rupees left, and that if he had not purchased any books he would have had 20 rupees left. How much did he spend in the purchase of stationery?

7. Of three horses, the first is worth 175 rupees, the second, 25 rupees more than the first, and the third, 50 rupees more than the first and the second together. What is the price of the third horse?

8. Given that the beginning of the year 1799 of the Saka era corresponds with the year 1877 of the Christian era; find the year of the Christian era when the Saka era commenced.

III.

1. Define the terms Product, Factor, and Power. Shew that Multiplication is only a concise method of Addition.

2. Shew that the local value of every figure after the units' figure in any number expressed in the Common System of Notation, has some power of 10 as one of its factors.

3. Can you multiply one concrete number by another? If not, give your reason.

4. Shew that in Multiplication, the order of the factors is immaterial, so far as the numerical value of the product is concerned.

5. In a garden there are 15 rows of trees, in each row there are 18 trees, and in each tree, 60 fruits. How many fruits are there in all?

6. If the price of a single copy of a book is 5 rupees, what is the price of 64 copies, and what the price of 6 times 64 copies?

7. If you can have 9 mangoes for 1 rupee, how many of the same quality can you have for 16 rupees?

8. There are 4 pice in an anna, 16 annas in a rupee, and 16 rupees in a gold mohur. How many pice are there in one rupee? and how many in one gold mohur?

IV.

1. Define the terms Dividend and Divisor. What are the two different meanings which the Quotient can have?

2. Can you divide one concrete number by another? If you can, what is the meaning of the quotient? Can you divide an abstract number by a concrete number?

3. Shew that

$$\text{dividend} - \text{remainder} = \text{divisor} \times \text{quotient}.$$

4. In Division by factors, how do you get the true remainder?

5. How many revolutions will the wheel of a carriage, 13 feet in circumference, make, in going over one-fourth part of a mile, there being 1760 yards in a mile, and 3 feet in a yard?

6. If 55 copies of a book can be had for 275 rupees, what is the price of a copy?

7. If 9 mangoes can be had for 1 rupee, what is the price of 117 mangoes?

8. If 192 mangoes can be had for 16 rupees, how many can be had for a rupee, and how many for 12 rupees?

V.

1. What do you mean by the Measure of a number. and what by the Greatest Common Measure of two or more numbers?

2. Shew that when two numbers are divided by their G. C. M., the quotients are prime to each other.

3. Shew that the number of measures of any number must be limited, but the number of its multiples is unlimited.

4. Shew that when the L. C. M. of two numbers is divided by each of those numbers, the quotients are prime to each other.

5. A gentleman has 24 rupees in his pocket, and he wishes to distribute the sum amongst the poor, in such a manner that each man shall receive the same number of rupees. In how many ways can he make the distribution?

6. *A* offers to distribute 36 pice amongst a number of beggars in such a manner that each shall receive the same number of pice; and *B* offers to distribute 24 pice amongst the same number in the same way. What is the largest number of beggars amongst whom the distribution can be made, and how many pice will each receive?

7. What is the least number of rupees that can be sorted in groups of 3, 4, 5, or 6 each?

8. Find the least number that is exactly divisible by the first nine odd numbers.

VI.

1. What is an Expression? and what is an Equation?

2. In simplifying an expression, what is the order in which you must proceed?

3. Simplify

$68 \div (3 \times 4 + 5) + 16 \times (20 \div 2 \times 3 - 30) - (4 - 8 \div 4)$.

4. Find the value of

$2 \times 3 \times 4 - 5 \times (21 - 4 \times 5) + 5 - 6 \div (32 - 3 \times 10)$.

5. Find the difference between

$6 \times (7 + 8) - 9 + 10 \div (11 - 2 \times 3) \times 12$,
and $7 \times (8 + 9) - 10 + 11 \div (11 - 2 \times 5) \times 13$.

6. Simplify

$10 \times 12 - (13 \times 3 - 19 \times 2) \div (3 \times 7 - 4 \times 5) + 60$.

7. Divide $22 \div 2 - (2 \times 3 + 5) + 656 - (4 \times 8 - 16 \div 2)$
by $20 \div 2 - (2 \times 5 - 5) + 56 - (64 \div 2 + 3)$.

8. Find the value of

$(3 - 2)^4 + (4 \times 2 - 18 \div 3)^4 - (66 \div 11 - 10 \div 5)^2$.

VII.

1. Shew that the local value of every significant digit after the units' figure in any number is a multiple of some power of ten.

2. Shew that a number is multiplied by any power of ten by affixing to its right a number of ciphers equal to the index of the power.

3. A woman brought a certain number of oranges to the market for sale. She sold 20 oranges to *A*; to *B*, 15 more than to *A*; and to *C*, 22 less than the number sold to *A* and *B* together; and she found that had she sold 10 more to each of the three persons *A*, *B*, and *C*, there would have remained only 3 oranges unsold. How many oranges were there at first?

4. A boy read 30 pages of a book in one week; in the next week he read 5 pages more than in the first; in the third week, 6 pages more than in the second; and in the fourth week as many pages as in the first two weeks together; and he

found that had he read 5 pages less every week, he would have gone through exactly one-third of the book. How many pages were there in the book ?

5. A horse and a carriage are together worth 1860 rupees, and the carriage is worth five times as much as the horse. What is the price of each ?

6. Two horses are together worth 850 rupees, and one of them is worth 250 rupees more than the other. What is the price of each horse ?

7. A boy spends 15 rupees in the purchase of books and fruits, the price of the books being four times as much as that of the fruits. How much are the books worth ?

8. A gentleman divides 1500 rupees amongst his 3 sons and 4 daughters, giving to each son twice as much as to each daughter. How much does each son get ?

VIII.

1. How can you ascertain by inspection whether a number is divisible by 4 ?

2. Shew that a number is divisible by 9 if the sum of its digits is divisible by 9.

3. A man dies leaving 5 sons and 2 daughters. His property is worth 36192 rupees. What is the value of the property inherited by each son, supposing a son's share to be twice as much as that of a daughter ?

4. Given that the quotient is 3025 and the dividend 36300, find the divisor.

5. What number multiplied by 627 will produce 11913 ? and what number added to the former will produce the latter ?

6. A father at the age of 60 is twice as old as his eldest son, and four times as old as the youngest. Find the difference between the ages of these sons.

7. In a certain town, it is found that the death rate is 3 per cent. and the birth rate 5 per cent. of the population at the beginning of the year. Supposing the population at the beginning of the year to be 20,000, what will be the population at the end of the same ?

8. The product of two numbers is 864, and their L. C. M. is 72. Find their G. C. M.

CHAPTER II.

THE FUNDAMENTAL OPERATIONS WITH ABSTRACT FRACTIONS.

INTRODUCTION.

78. Besides *integers* or *multiples* of unity which we considered in the preceding Chapter, we have frequently to deal with *fractions* or *parts* of unity. Thus, if we have to divide 6 units into 4 equal parts, we get as the result the quotient 1, and the remainder 2, which means that 6 units divided into 4 equal parts give 1 unit for each part, and there remain besides 2 units which cannot be divided into 4 equal parts in integers. To complete the division, we must *break* these 2 units each into *two equal parts* or *halves*, thus getting 4 halves, which divided into 4 equal parts give a *half* for each part ; so that the complete quotient is *one and a half ; i. e.*, 6 units divided into 4 equal parts give a unit and a half for each part.

The word *fraction* (from the Latin *frangere*, to break) literally means a broken part.

79. By *multiplying* the *primary* unit *one* by 1, 2, 3, 4, &c., we obtain an unlimited series of numbers, *viz.*, 1, 2, 3, 4, &c., which contains all possible integers, and no other numbers. If we *divide* the *primary* unit *one* into 2, 3, 4, &c. equal parts, we obtain an unlimited series of *secondary* units or parts of the primary unit, *viz.*, *one-half, one-third, one-fourth, &c.*, by multiplying each of which by 1, 2, 3, 4, &c., we obtain an unlimited number of unlimited series, *viz.*,

One-half, two-halves, three-halves, four-halves, &c. ;
One-third, two-thirds, three-thirds, four-thirds, &c. ;
One-fourth, two-fourths, three-fourths, four-fourths, &c. ;
&c., &c., &c. ;

which contain all possible numbers, whether integral or fractional. Thus, the second term of the first series, *two-halves,*

is really the integer 1; so its fourth term *four-halves* is really the integer 2; and so on; and that these series give all possible fractional numbers or parts of unity, and collections of parts of unity not composing entire units, is obvious.

Again, as integers are expressed in the Common System of Notation in an ascending scale of *tens, hundreds, &c.*, so for fractions, we may, by dividing the primary unit *one* into 10, 100, &c. equal parts, and thus getting a series of *secondary* or fractional units, *viz., one-tenth, one-hundredth, &c.*, obtain by the repetition of these last a descending scale of *tenths, hundredths, &c.* Here as we divide the primary unit into 10,100, &c. equal parts to obtain our secondary units, it is not evident that in this mode, all possible fractions can be expressed. Whether every possible fraction can be really expressed in this mode, will be considered in Sections VII and XII of this Chapter.

Def. Fractions of the former class, that is, fractions consisting of parts of unity obtained by dividing unity into *any* number of equal parts, are called Vulgar Fractions or *ordinary* fractions, to distinguish them from fractions of the latter class, that is, fractions consisting of parts of unity obtained by dividing unity into *ten, hundred, &c.* equal parts, which are called Decimal Fractions or Decimals, from their having *ten* (which is *decem* in Latin) for their basis.

80. Each of the above systems of fractions has its peculiar advantages and disadvantages, and we shall consider each in a separate Division of this Chapter.

DIVISION I. VULGAR FRACTIONS.

SECTION I. NOTATION AND NUMERATION OF FRACTIONS.

81. We have seen that a fraction is an integral number of parts of unity, which parts result from the division of unity by some integer. Thus the fraction *three-fourths* is one in which 3 parts of unity are taken, such parts resulting from the division of unity by 4.

When the number of parts composing a fraction is a multiple of the number of parts into which unity is divided, the fraction, though a fraction in *form*, is *really* an integer. For the fraction really consists of *all* the parts into which unity is divided, *i. e.*, of the *entire* unit itself, taken a certain number of times.

Thus, the fraction *six-thirds* means *six* times the *third* part of unity, *i. e.*, *twice three* times the *third* part of unity, *i.e.*, *two* units.

When the number of parts composing a fraction is greater than the number of parts into which unity is divided, any portion of the former which is equal to, or is a multiple of, the latter, may be separated from the rest, and expressed in the form of an integer. Thus, the fraction *seven-thirds* may be separated into two portions, *six-thirds* and *one-third*, whereof the former is equal to the integer *two*, and thus the whole fraction is equal to *two* and *one-third.*

DEF. In any fraction, the *number* of parts into which unity is divided is called the DENOMINATOR of that fraction, and the *number* of parts taken is called the NUMERATOR; and both these are called the TERMS of the fraction.

The propriety of these names is evident. The former is called the *denominator*, because it shews the *denomination* of the secondary units or parts of which the fraction is composed; and the latter is called the *numerator*, because it shews the *number* of such parts or secondary units intended to be taken.

A fraction is *named* by naming the numerator or the *number* of parts of unity taken, and after it the *ordinal* of the

denominator or the *number* by which unity is divided to give one of such parts.

Thus where unity is divided into *seven* equal parts, and *three* of such parts are taken, the fraction is named *three-sevenths.*

82. A fraction is *represented* by writing the numerator and the denominator, which are integers, according to the Common System of Notation, the former above the latter, with a line between.

Thus, the fraction *three-fifths* is written $\frac{3}{5}$;
.............*seven-sixteenths*..........$\frac{7}{16}$;
&c. &c. &c.

Hence the secondary or fractional units will be written $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.

Def. The fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. are called the Reciprocals of 2, 3, 4, 5, &c.

83. This mode of Notation is obviously *complete.* For, as in the Common System of Notation we can express every integral numerator and denominator, we can express every fraction, whatever its numerator and denominator may be, in this mode.

It is also a *convenient* mode. For, as in naming a fraction, such as *three-fourths,* we *name* first the numerator, and then the denominator, so in our Notation, we *write* in the same order, first the numerator, and then the denominator.

84. It now remains to be seen whether this Notation of fractions is *consistent* with our *previous conventions.*

For, the symbol $\frac{\text{numerator}}{\text{denominator}}$ by which we have expressed fractions, not being a *new* symbol, but being one which *has already been used* in Art. 14 to indicate the division of the upper number by the lower, we must shew that a fraction also denotes the division of the numerator by the denominator, in order to establish the consistency of our Notation, and to be able to use it without confusion, such as would result from the expression of *really different things* by the *same symbol* in the same science.

Now to shew that a fraction indicates the division of the numerator by the denominator, or in other words, is equal to the quotient arising from the division of the numerator by the denominator, take any fraction *four-fifths.*

This fraction indicates that unity is divided into 5 equal parts of which 4 parts are taken; *i. e.*, it denotes 4 *times the fifth part of* 1 *unit.*

But when 4 units are divided into 5 equal parts, then also, each part is equal to 4 *times the fifth part of* 1 *unit.*

Thus the fraction *four-fifths* means the same thing as 4 units divided into 5 equal parts. And so for any other fraction.

Hence a fraction denotes *the magnitude of each part* of the numerator when it is divided into the number of equal parts indicated by the denominator. And this is one of the meanings attached to the term quotient. (Art. 50.)

Next let us see how far a fraction has the other meaning of the term, *i. e.*, indicates the *number of times* that the numerator contains the denominator.

Taking the same fraction, *four-fifths*, we see that 5, being 5 times as large as 1, is contained in 1 *to the extent only of its fifth part*, and is contained in 4 *to the extent of* 4 *times its fifth part*, or in other words, *to the extent of its four-fifths.* Now, as when one number is contained in another to the extent of its *double, triple, &c.*, we say that it is contained in that other *two times, three times, &c.;* so when one number is contained in another to the extent of its *half, third,* &c. part, we can by a *stretch of language* say, that it is contained in that other *one-half, one-third,* &c. *part of a time.* And in this sense, as 5 is contained in 4 to the extent of its *four-fifths*, we can say that 5 is contained in 4 *four-fifths of a time.*

Thus the fraction *four-fifths* represents the *number of times*, or rather, the *number of parts of a time*, that 5 is contained in 4. And so for any other fraction.

Hence a fraction also denotes *the number of times* that the denominator is contained in the numerator.

This establishes the consistency of our Notation.

The above considerations enable us to express *numerically* the *complete* quotient arising from the Division of integers, in cases in which the dividend is not exactly divisible by the divisor.

Thus, take as an example $19 \div 3$.

The result of the division in this case, expressed in integers, is the quotient 6 and the remainder 1, which means that one part of 19, namely, 18, contains 3, 6 times, and the other part 1 not containing 3 any integral number of times is represented as a part of the dividend that *remains* to be divided by 3. This part, 1, however, contains 3 *one-third part of a time*; and therefore by a *stretch of language* we can say that 3 is contained in the entire dividend 19, 6 times together with the *third* part of a time, or shortly, $6 + \frac{1}{3}$ times; so that $6 + \frac{1}{3}$ (written also $6\frac{1}{3}$) is the *complete* quotient arising from the division.

It may be observed that the same *symbolical* expression for the quotient may be deduced without reference to the Notation of fractions. Thus,

$$\frac{19}{3} = \frac{18+1}{3} = \frac{18}{3} + \frac{1}{3} = 6 + \frac{1}{3}.$$

85. DEFS. A fraction of which no part is expressed in an integral form, and whose numerator and denominator are integers, is called a SIMPLE FRACTION.

Thus, $\frac{3}{5}$, $\frac{7}{6}$, $\frac{3}{10}$ are simple fractions.

A fraction whose numerator is less than the denominator, is called a PROPER FRACTION.

Thus, $\frac{4}{5}$, $\frac{3}{7}$, $\frac{2}{9}$ are proper fractions.

A fraction whose numerator is not less than the denominator, is called an IMPROPER FRACTION.

Thus, $\frac{7}{5}$, $\frac{8}{8}$, $\frac{13}{5}$ are improper fractions.

A number expressed partly in an integral and partly in a fractional form is called a MIXED NUMBER.

Thus, $1\frac{1}{3}$, $3\frac{4}{5}$, $7\frac{2}{3}$ are mixed numbers.

86. PROP. I. Any integer may be represented in the form of a fraction by making that integer the numerator, and unity the denominator, or by making the product of that integer

and any other integer the numerator, and that other integer the denominator.

Thus, 5 may be represented as $\frac{5}{1}$; or $\frac{5 \times 2}{2}$, *i. e.*, $\frac{10}{2}$; or $\frac{5 \times 3}{3}$, *i. e.*, $\frac{15}{3}$; &c.

For $\frac{5}{1}$ means either that 5 is divided into 1 part, *i. e.*, kept entire; or, that unity is divided into 1 part or kept entire, and 5 such parts are taken; *i. e.*, in either sense it means 5 units: $\therefore \frac{5}{1} = 5$.

So $\frac{10}{2}$ means either that 10 is divided into 2 equal parts and one of such parts is taken; or, that unity is divided into 2 equal parts or halves, and 10 such parts are taken; *i. e.*, in either sense it means 5 units: $\therefore \frac{10}{2} = 5$. Similarly $\frac{15}{3} = 5$. And so on.

Prop. II. Every fraction whose numerator is a multiple of the denominator, is really an integer equal to the quotient arising from the division of the numerator by the denominator.

Thus, $\frac{10}{5} = 2$.

87. Besides the kinds of fraction defined in Art. 85, there are two other kinds which are defined below.

Defs. A Compound Fraction is a fraction of a fraction.

Thus, $\frac{2}{3}$ of $\frac{3}{4}$ is a compound fraction.

A Complex Fraction is a fraction having a mixed number or a fraction for its numerator or denominator or both.

Thus, $\frac{2\frac{1}{2}}{3}$, $\frac{4}{5\frac{6}{7}}$, $\frac{\frac{2}{5}}{6}$, $\frac{8}{\frac{3}{4}}$, $\frac{2\frac{1}{2}}{3\frac{3}{5}}$, $\frac{\frac{5}{6}}{\frac{8}{15}}$ are complex fractions.

The meanings of these two forms of fractions are not evident from the considerations in Arts. 78 to 81. We must therefore *interpret* these forms, *i. e.*, see what they mean.

1st. *Compound Fractions.* Take as an example $\frac{3}{4}$ of $\frac{4}{5}$.

Then $\because$ to obtain $\frac{3}{4}$, or $\frac{3}{4}$ of 1, we divide 1 into 4 equal parts, and take 3 of such parts, $\therefore$ to obtain $\frac{3}{4}$ of $\frac{4}{5}$, we must *divide*

$\frac{4}{5}$, *or the secondary units that compose* $\frac{4}{5}$, *into* 4 *equal parts, and take* 3 *of such parts.* And this is the meaning of $\frac{3}{4}$ of $\frac{4}{5}$.

How this division of the secondary units into parts and the multiplication of those parts are to be effected, in other words, how the value of a compound fraction is to be expressed in the form of a simple fraction, will be shewn in Art. 92.

2nd. *Complex Fractions.* Take as an example $\frac{2\frac{3}{4}}{5\frac{3}{5}}$.

Then $\because$ $\frac{\text{one number}}{\text{another number}}$ means the former divided by the latter,

$\therefore$ $\frac{2\frac{3}{4}}{5\frac{3}{5}}$ will mean $2\frac{3}{4}$ *divided by* $5\frac{3}{5}$, *i. e., the secondary units of which the former is composed divided by those of which the latter is composed.*

How this division is to be effected, in other words, how the value of a complex fraction is to be expressed in the form of a simple fraction, will be shewn in Art. 93.

Ex. VIII.

1. Express as fractions—

(1) The integer 5 having 2, 3, and 4 for the denominator.
(2) ... ... 9 ... 3, 4, ... 5.........................
(3) ... ... 7 ... 6, 7, ... 8.........................
(4) ... ... 8 ... 3, 5, ... 7.........................
(5) ... ... 16 ... 11, 12, ... 13.........................
(6) ... ... 25 ... 4, 8, ... 16.........................

2. Convert the following fractions into their equivalent integers :—

(1) $\frac{16}{8}$. (2) $\frac{9}{3}$. (3) $\frac{27}{9}$. (4) $\frac{28}{7}$. (5) $\frac{38}{19}$. (6) $\frac{121}{11}$.
(7) $\frac{256}{8}$. (8) $\frac{171}{9}$. (9) $\frac{324}{9}$. (10) $\frac{1728}{12}$. (11) $\frac{360}{60}$. (12) $\frac{210}{70}$.

SECTION II. TRANSFORMATION OF FRACTIONS.

88. We can *transform* one fraction into another, *i. e.*, we can change the *form* of a fraction without altering its value. In effecting this, the Propositions in the following Article will be of use.

89. Prop. I. The value of a fraction is not altered if we multiply or divide both the numerator and the denominator by the same number.

Thus, take any fraction $\frac{4}{6}$. Here unity is divided into 6 equal parts and 4 of such parts are taken.

Multiplying both its terms by 2, we have $\frac{4 \times 2}{6 \times 2} = \frac{8}{12}$, which means that unity is divided into 12 equal parts whereof 8 are taken. Now although each part in the latter case being a *twelfth* part of unity is only *half* as large as each part in the former case, namely a *sixth*, yet as in the latter case we take 8 parts, which are *twice* as many as the parts taken in the former case, in fact we take just as much of unity now as we took before.

Hence $\frac{4}{6} = \frac{8}{12}$.

Again, dividing the terms by 2, we have $\frac{4 \div 2}{6 \div 2} = \frac{2}{3}$ which means that unity is divided into 3 equal parts whereof 2 are taken. Now although here each part being *one-third* is *twice* as large as *one-sixth*, yet as we now take only 2 parts, which are *half* as many as the parts taken before, in fact we take just as much of unity as we took before.

Hence $\frac{4}{6} = \frac{2}{3}$.

This may be illustrated thus:—

A e E c C f F d D g G b B

Let the line A B represent unity.

Let A B be divided into 6 equal parts AE, EC, CF, FD, DG, and GB; then 4 of these parts taken together, *i. e.*, AE + EC + CF + FD make up AD.

Thus $\frac{4}{6}$ of AB = AD.

Again, let AB be divided into 12 equal parts by bisecting or dividing into 2 equal parts each of the above 6 parts, so that we have Ae, eE, Ec, cC, Cf, fF, Fd, dD, Dg, gG, Gb, and bB, for the 12 parts; then 8 of these parts taken together, *i. e.*, Ae + eE + Ec + cC + Cf + fF + Fd + dD make up AD.

Thus $\frac{8}{12}$ of AB = AD = $\frac{4}{6}$ of AB.

Lastly, let AB be divided into 3 equal parts by taking AE + EC, CF + FD, and DG + GB, to be the parts; then 2 of these parts together, *i. e.*, (AE + EC) + (CF + FD) make up AD.

Thus $\frac{2}{3}$ of AB = AD = $\frac{4}{6}$ of AB.

Hence, *if the numerator and the denominator contain any common factor, it may be struck out without altering the value of the fraction.*

For, to strike out a factor of any number is to divide the number by that factor.

Thus $\frac{4}{6} = \dfrac{2 \times 2}{3 \times 2} = \dfrac{2 \times 2 \div 2}{3 \times 2 \div 2} = \frac{2}{3}$.

PROP. II. A fraction is multiplied or divided by an integer by multiplying its numerator or denominator by that integer.

Thus, let it be required to multiply $\frac{4}{8}$ by 3.

Since $\frac{4}{8} \times 3$ means that the secondary units or parts of which the fraction $\frac{4}{8}$ is composed are to be multiplied by 3, the product will be obtained by multiplying the numerator 4, which denotes the number of parts composing the fraction, by 3, and keeping the denominator unchanged; so that

$$\frac{4}{8} \times 3 = \frac{4 \times 3}{8} = \frac{12}{8}.$$

Again, let it be required to divide $\frac{4}{8}$ by 3.

Since $\frac{4}{8} \div 3$ means that the secondary units or parts of which the fraction $\frac{4}{8}$ is composed are to be divided into 3 equal parts, the quotient will be obtained by making each part *one-third* of what it is, and this is done by dividing the primary unit into *thrice* as many parts as before, *i. e.*, into 3×8 parts, and keeping unchanged the number of parts taken; so that

$$\frac{4}{8} \div 3 = \frac{4}{3 \times 8} = \frac{4}{24}.$$

PROP. III. A fraction is multiplied or divided by an integer by dividing its denominator or numerator by that integer.

Thus $\frac{2}{6} \times 2 = \dfrac{2 \times 2}{6}$ (by Prop. II) $= \dfrac{2 \times 2 \div 2}{6 \div 2}$ (by Prop. I) $= \frac{2}{3}$.

So $\frac{2}{6} \div 2 = \dfrac{2}{6 \times 2}$ (by Prop. II) $= \dfrac{2 \div 2}{6 \times 2 \div 2}$ (by Prop. I) $= \frac{1}{6}$.

PROP. IV. A proper fraction is less than unity; and an improper fraction is greater than, or equal to, unity, according as the numerator is greater than, or equal to, the denominator.

For, in a proper fraction, the numerator being less than the denominator, we take fewer parts than those into which unity is divided, *i. e.*, something less than unity.

Again, in an improper fraction, when the numerator exceeds the denominator, we take more parts than those into which unity is divided, *i. e.*, something more than unity.

Lastly, when the numerator is equal to the denominator, we take as many parts as those into which unity is divided, *i. e.*, exactly the whole of unity.

90. *To reduce an improper fraction to a mixed number.*

RULE. Divide the numerator by the denominator; put the quotient for the integral part, and the remainder for the numerator of the fractional part, and the former denominator for its denominator.

The *reason for the Rule* will be clear from the following Example.

Ex. Reduce $\frac{19}{5}$ to a mixed number.

We have $\frac{19}{5} = 19 \div 5 = 3$ together with 4 remaining to be divided by $5 = 3 + \frac{4}{5} = 3\frac{4}{5}$.

91. *To reduce a mixed number to a simple fraction.*

RULE. To the numerator add the product of the denominator and the integral part for the new numerator, and put the denominator below.

The *reason for the Rule* will appear from the Example below.

Ex. Reduce $6\frac{2}{3}$ to an improper fraction.

We have $6\frac{2}{3} = 6 + \frac{2}{3} = \frac{6 \times 3}{3} + \frac{2}{3}$ (Art. 86, Prop. I)

$= \frac{18}{3} + \frac{2}{3} = 18$ *thirds* of unity + 2 *thirds* of unity

$= (18 + 2)$ *thirds* of unity

$18 + 2$

$= \frac{20}{3}.$

92. *To reduce a compound fraction to a simple fraction.*

RULE. After having reduced (if necessary) the component fractions to simple ones, write down the product of the several numerators for the new numerator, and the product of the several denominators for the new denominator, and cancel or strike out all the factors that are common to the numerator and the denominator.

The *reason for the Rule* will appear from the Example below.

Ex. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{8}$ to a simple fraction.

We have $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{8} = \frac{2}{3}$ of $(\frac{3}{4}$ of $\frac{4}{8})$.

Now $\frac{3}{4}$ of $\frac{4}{8} = (\frac{4}{8} \div 4) \times 3$ (Art. 87)

$$= \frac{4}{8 \times 4} \times 3 = \frac{4 \times 3}{8 \times 3} \text{ (Art. 89, Prop. II)};$$

$$\therefore \quad \tfrac{2}{3} \text{ of } \tfrac{3}{4} \text{ of } \tfrac{4}{8} = \tfrac{2}{3} \text{ of } \frac{4 \times 3}{8 \times 4}$$

$$= \frac{4 \times 3 \times 2}{8 \times 4 \times 3} \text{ (in the same way)}$$

$$= \tfrac{1}{4} \text{ (after striking out common factors).}$$

93. *To reduce a complex fraction to a simple fraction.*

RULE. After having reduced (if necessary) the numerator and the denominator to the form of simple fractions, put the product of the numerator of the upper fraction and the denominator of the lower for the new numerator, and the product of the denominator of the upper fraction and the numerator of the lower for the new denominator, and strike out all the factors that are common to the numerator and the denominator.

The *reason for the Rule* will appear from the Example below.

Ex. Reduce $\frac{2\frac{1}{3}}{1\frac{2}{5}}$ to a simple fraction.

We have $\frac{2\frac{1}{3}}{1\frac{2}{5}} = \frac{\frac{7}{3}}{\frac{7}{5}}$ (Art. 91)

$$= \tfrac{7}{3} \div \tfrac{7}{5} \text{ (Art. 87).}$$

Now $\frac{7}{3} = \frac{7 \times 5}{3 \times 5} = \frac{7 \times 5}{15}$

$= (7 \times 5)$ secondary units whereof each is $\frac{1}{15}$th of 1.

So $\frac{7}{5} = \frac{7 \times 3}{15}$

$= (7 \times 3)$..

$\therefore \quad \frac{7}{3} \div \frac{7}{5} =$ the number of times that (7×3) secondary units each being $\frac{1}{15}$ of 1 are contained in (7×5) secondary units of the same value

$= (7 \times 5) \div (7 \times 3)$

$=$ the fraction $\dfrac{7 \times 5}{7 \times 3}$ (Art. 84.)

$= \frac{5}{3}$ (after striking out the common factor 7).

94. When the numerator and the denominator of a simple fraction contain any common factor, we have seen (Art. 89, Prop. I) that it may be struck out or canceled, without altering the value of the fraction, and the fraction may thus be expressed by a numerator and a denominator which are *less* than what they were before. This *lowering* of the *terms* of a fraction is, however, not possible when they are prime to each other. Accordingly we have the following Definition.

Def. A simple fraction is said to be in its Lowest Terms when its numerator and denominator are prime to each other.

95. *To reduce a simple fraction to its lowest terms.*

Rule. After striking out from the numerator and the denominator any common factors that may be found by inspection, divide the resulting numerator and denominator by their G. C. M., and write the corresponding quotients as the new numerator and the new denominator.

The *reason for the Rule* is evident. The division of both the terms by their G. C. M. does not alter the value of the fraction, and the new numerator and the new denominator are prime to each other by Art. 68.

Ex. 1. Reduce $\frac{936}{2368}$ to its lowest terms.

We have $\frac{936}{2368} = \frac{117}{296}$ (by striking out the factor 8).

Now the G. C. M. of 117 and 296 is 1, or in other words 117 and 296 are prime to each other.

Hence $\frac{936}{2368}$ reduced to its lowest terms

$= \frac{117}{296}$.

We may, without proceeding to find the G. C. M. of 117 and 296, say that they are prime to each other. For $117 = 9 \times 13 = 3 \times 3 \times 13$, and neither 3 nor 13 is a factor of 296.

Ex. 2. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{8}{9}$ to a simple fraction in its lowest terms.

We have $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{8}{9} = \dfrac{2 \times 3 \times 8}{3 \times 4 \times 9} = \dfrac{2 \times 3 \times 2 \times 4}{3 \times 4 \times 9}$

$$= \frac{2 \times 2}{9} = \frac{4}{9}.$$

96. *To reduce fractions to their equivalent ones having the least common denominator.*

Rule. Having reduced the fractions to simple fractions in their lowest terms, find the L. C. M. of the resulting denominators, and write it as the new denominator of each.

Divide it by each denominator, multiply the quotient by the corresponding numerator, and put down the product as the corresponding new numerator.

The *reason for the Rule* is clear. The fractions being first reduced to their lowest terms, have their denominators the least possible; and the only way in which we can reduce these fractions to the required form without altering their values, is by multiplying both the terms of each by the same quantity such that the resulting denominator will in each case be the same, and the least possible.

Now $\because$ the required least common denominator results from the multiplication of each denominator by some integer, it must be the L. C. M. of the denominators; and the number by which the terms of each fraction must be multiplied, is the quotient arising from the division of this L. C. M. by the corresponding denominator. By this process, the resulting denominator will be the same in each case, being the L. C. M. of the denominators.

Ex. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{4}{6}$, and $\frac{1}{8}$ to their equivalent fractions, with the least common denominator.

Reducing the fractions to their lowest terms we have

$$\frac{1}{2}, \frac{3}{4}, \frac{2}{3}, \text{ and } \frac{1}{8}.$$

The L. C. M. of the denominators is 12, and the several quotients are 6, 3, 4, 4; so that the required fractions are

$$\frac{1\times 6}{2\times 6},\quad \frac{3\times 3}{4\times 3},\quad \frac{2\times 4}{3\times 4},\ \text{and}\quad \frac{1\times 4}{3\times 4},$$

or $\frac{6}{12}$, $\frac{9}{12}$, $\frac{8}{12}$, and $\frac{4}{12}$.

97. Prop. I. Of two fractions having the same denominator, the one with the greater numerator is the greater.

For, the denominators being the same, unity is divided into the same number of parts in both cases, and ∴ the magnitude of the parts is the same in both; and hence that fraction is the greater which is composed of the larger number of parts, *i. e.*, which has the greater numerator.

Prop. II. Of two fractions having the same numerator, the one with the smaller denominator is the greater.

For, the same number of parts being taken in both cases, that fraction is the greater in which the magnitude of each part is the greater, *i. e.*, in which unity is divided into the smaller number of parts, *i. e.*, in which the denominator is the smaller.

98. By the preceding Article, we can compare the values of fractions by reducing them to their equivalent ones with the least common denominator, and then comparing their numerators; or by comparing their denominators, if they have the same numerator.

Ex. 1. Compare the values of $\frac{1}{2}$, $\frac{5}{6}$, $\frac{3}{8}$, and $\frac{2}{3}$.

The given fractions reduced to their equivalent ones with the least common denominator become $\frac{12}{24}$, $\frac{20}{24}$, $\frac{9}{24}$, and $\frac{16}{24}$, and these in the descending order are

$\frac{20}{24}$, $\frac{16}{24}$, $\frac{12}{24}$, and $\frac{9}{24}$;

∴ the given fractions similarly arranged are

$\frac{5}{6}$, $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{3}{8}$.

Ex. 2. Compare the values of $\frac{1}{3}$, $\frac{2}{10}$, $\frac{3}{12}$, and $\frac{5}{30}$.

Reduced to their lowest terms, the fractions are

$\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{4}$, and $\frac{1}{6}$.

By Art. 97, Prop. II, these in the descending order are

$\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$;

∴ the given fractions in the same order are

$\frac{1}{3}$, $\frac{3}{12}$, $\frac{2}{10}$, and $\frac{5}{30}$.

Ex. IX.

1. Express the following improper fractions as mixed or whole numbers :—

(1) $\frac{3}{2}$. (2) $\frac{5}{3}$. (3) $\frac{10}{5}$. (4) $\frac{16}{15}$.
(5) $\frac{18}{7}$. (6) $\frac{24}{3}$. (7) $\frac{103}{13}$. (8) $\frac{602}{26}$.
(9) $\frac{456}{12}$. (10) $\frac{123}{45}$. (11) $\frac{789}{56}$. (12) $\frac{1000}{99}$.

2. Reduce the following mixed numbers to the form of simple fractions :—

(1) $1\frac{1}{3}$. (2) $3\frac{2}{5}$. (3) $5\frac{3}{7}$. (4) $8\frac{9}{10}$.
(5) $11\frac{11}{12}$. (6) $13\frac{9}{19}$. (7) $29\frac{3}{17}$. (8) $19\frac{5}{8}$.
(9) $16\frac{6}{11}$. (10) $121\frac{19}{32}$. (11) $100\frac{10}{11}$. (12) $1000\frac{1}{100}$.

3. Reduce the following compound fractions to simple fractions in their lowest terms :—

(1) $\frac{1}{3}$ of $\frac{3}{4}$. (2) $\frac{1}{4}$ of $\frac{2}{3}$. (3) $\frac{2}{5}$ of $\frac{3}{10}$.
(4) $\frac{1}{9}$ of $\frac{3}{5}$. (5) $\frac{3}{2}$ of $\frac{6}{8}$. (6) $\frac{10}{11}$ of $\frac{121}{100}$.
(7) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$. (8) $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$. (9) $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$.

(10) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$.
(11) $\frac{3}{4}$ of $\frac{6}{15}$ of $\frac{8}{9}$ of $\frac{5}{6}$.
(12) $3\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{1}{9}$ of $\frac{63}{21}$ of $9\frac{3}{5}$ of $\frac{7}{12}$ of $\frac{9}{112}$.
(13) $\frac{7}{8}$ of $\frac{120}{136}$ of $\frac{9}{25}$ of $3\frac{3}{11}$ of $\frac{2}{9}$.
(14) $2\frac{34}{100}$ of $\frac{4}{13}$ of $\frac{5}{9}$ of $\frac{31}{32}$ of $\frac{16}{62}$.
(15) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$.
(16) $1\frac{1}{2}$ of $1\frac{1}{3}$ of $1\frac{1}{4}$ of $1\frac{1}{5}$ of $1\frac{1}{6}$ of $1\frac{1}{7}$ of $1\frac{1}{8}$ of $1\frac{1}{9}$.
(17) $\frac{11}{12}$ of $\frac{21}{22}$ of $\frac{31}{32}$ of $\frac{41}{42}$ of $\frac{51}{52}$ of $\frac{61}{62}$.
(18) $\frac{12}{21}$ of $\frac{34}{43}$ of $\frac{56}{65}$ of $\frac{78}{87}$ of $\frac{90}{9}$.

4. Reduce the following complex fractions to simple fractions in their lowest terms:—

(1) $\dfrac{2\frac{2}{3}}{3\frac{3}{4}}$. (2) $\dfrac{1\frac{1}{2}}{2\frac{1}{4}}$. (3) $\dfrac{5\frac{1}{4}}{2\frac{1}{3}}$. (4) $\dfrac{7\frac{1}{7}}{4\frac{1}{6}}$.

(5) $\dfrac{28}{4\frac{1}{5}}$. (6) $\dfrac{21\frac{1}{4}}{17}$. (7) $\dfrac{33\frac{1}{3}}{16\frac{2}{3}}$. (8) $\dfrac{27\frac{1}{2}}{2\frac{3}{4}}$.

(9) $\dfrac{\frac{1}{3}\text{ of }\frac{1}{4}}{12}$. (10) $\dfrac{\frac{2}{3}\text{ of }\frac{3}{4}}{\frac{3}{4}\text{ of }\frac{4}{5}}$. (11) $\dfrac{\frac{1}{2}\text{ of }\frac{1}{3}}{\frac{1}{30}}$. (12) $\dfrac{8\frac{1}{10}}{\frac{3}{5}\text{ of }\frac{9}{10}}$.

(13) $\dfrac{\frac{1}{2}\text{ of }\frac{1}{3}\text{ of }\frac{1}{4}}{\frac{1}{2}\text{ of }\frac{1}{3}}$. (14) $\dfrac{\frac{2}{3}\text{ of }\frac{6}{7}\text{ of }\frac{8}{9}}{\frac{4}{5}\text{ of }\frac{5}{9}\text{ of }\frac{1}{2}}$.

(15) $\dfrac{2\frac{1}{2}\text{ of }3\frac{3}{5}}{1\frac{1}{2}\text{ of }2\frac{2}{3}}$. (16) $\dfrac{28\frac{1}{3}\text{ of }1\frac{1}{17}\text{ of }\frac{1}{30}}{18\frac{1}{3}\text{ of }1\frac{1}{5}\text{ of }\frac{1}{22}}$.

5. Reduce the following fractions to their lowest terms :—

(1) $\frac{9}{15}$. (2) $\frac{10}{22}$. (3) $\frac{16}{52}$. (4) $\frac{39}{91}$.
(5) $\frac{121}{990}$. (6) $\frac{361}{456}$. (7) $\frac{324}{288}$. (8) $\frac{225}{945}$.
(9) $\frac{169}{273}$. (10) $\frac{585}{780}$. (11) $\frac{195}{260}$. (12) $\frac{460}{690}$.
(13) $\frac{123}{456}$. (14) $\frac{789}{1011}$. (15) $\frac{135}{999}$. (16) $\frac{369}{963}$.
(17) $\frac{24}{642}$. (18) $\frac{248}{840}$. (19) $\frac{256}{720}$. (20) $\frac{700}{910}$.

6. Reduce the fractions in each of the following sets to equivalent fractions having the least common denominator :—

(1) $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. (2) $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$.
(3) $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{3}{10}$. (4) $\frac{5}{6}$, $\frac{2}{9}$, and $\frac{7}{12}$.
(5) $\frac{4}{15}$, $\frac{5}{18}$, and $\frac{6}{21}$. (6) $\frac{5}{12}$, $\frac{6}{18}$, and $\frac{7}{21}$.
(7) $\frac{3}{8}$, $\frac{4}{9}$, $\frac{5}{10}$, and $\frac{6}{12}$. (8) $\frac{2}{6}$, $\frac{3}{7}$, $\frac{4}{8}$, and $\frac{5}{9}$.
(9) $\frac{21}{22}$, $\frac{31}{33}$, $\frac{41}{44}$, and $\frac{1}{13}$. (10) $\frac{8}{10}$, $\frac{10}{12}$, $\frac{12}{14}$, and $\frac{14}{16}$.
(11) $\frac{54}{81}$, $\frac{50}{90}$, $\frac{49}{91}$, and $\frac{13}{14}$. (12) $\frac{11}{111}$, $\frac{22}{222}$, $\frac{33}{333}$, and $\frac{2}{3}$.
(13) $\frac{2}{2}$, $\frac{2}{3}$, $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$, $\frac{2}{7}$, $\frac{2}{8}$, and $\frac{2}{9}$.
(14) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, and $\frac{8}{9}$.
(15) $\frac{2}{2}$, $\frac{3}{4}$, $\frac{2}{6}$, $\frac{3}{8}$, and $\frac{3}{10}$.
(16) $\frac{1}{36}$, $\frac{2}{69}$, $\frac{3}{912}$, and $\frac{4}{1215}$.
(17) $\frac{1}{12}$, $\frac{2}{24}$, $\frac{4}{48}$, and $\frac{8}{816}$.
(18) $\frac{9}{10}$, $\frac{9}{16}$, $\frac{9}{100}$, $\frac{9}{160}$, and $\frac{9}{1000}$.

7. Compare the values of

(1) $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$. (2) $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{6}{8}$.
(3) $\frac{3}{5}$, $\frac{5}{6}$, and $\frac{7}{10}$. (4) $\frac{7}{8}$, $\frac{9}{15}$, and $\frac{23}{24}$.

(5) $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{6}{15}$, and $\frac{27}{18}$. (6) $\frac{3}{7}$ of $2\frac{1}{3}$, $\frac{2}{9}$ of $\frac{3}{8}$, and $\frac{29}{36}$.

(7) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, and $\frac{1}{9}$.

(8) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, and $\frac{8}{9}$.

(9) $\frac{5}{9}$, $\frac{4}{8}$, $\frac{3}{7}$, $\frac{2}{6}$, and $\frac{2}{5}$.

(10) $\frac{4}{9}$, $\frac{5}{8}$, $\frac{6}{7}$, $\frac{3}{6}$, $\frac{2}{5}$, and $\frac{1}{4}$.

(11) $\frac{12}{21}$, $\frac{22}{31}$, $\frac{32}{41}$, and $\frac{42}{51}$.

(12) $\frac{9}{20}$, $\frac{13}{25}$, $\frac{12}{273}$, $\frac{7}{12}$, and $\frac{7}{35}$.

(13) $\frac{12}{16}$, $\frac{23}{25}$, $\frac{22}{56}$, $\frac{28}{24}$, and $\frac{24}{128}$.

(14) $\frac{5}{13}$, $\frac{10}{24}$, $\frac{15}{21}$, and $\frac{25}{105}$.

(15) $\frac{2}{10}$, $\frac{22}{100}$, $\frac{90}{1000}$, and $\frac{222}{900}$.

SECTION III. ADDITION OF FRACTIONS.

99. Rule. Reduce compound and complex fractions to simple ones; improper fractions to mixed numbers, and all the proper fractions to their lowest terms.

Add together the integral parts of the summands, if any, as in Simple Addition.

Reduce the proper fractions in the several summands to their equivalent ones having the least common denominator; add together the new numerators of these fractions, and put down the result as the numerator of the sum, and the least common denominator as its denominator.

The resulting fraction, reduced if possible, together with the sum of the integral parts already found, will be the sum required.

Reason for the Rule. To add numbers together is to add together their integral parts and also their fractional parts; and to add fractions together is to add together the *numbers* of parts of unity that they contain when reduced to a common denominator. Thus, take the following Example :—

Ex. Add together $2\frac{1}{2}$, $\frac{1}{3}$ of $\frac{3}{5}$, $5\frac{2}{10}$, and $\frac{3}{8}$.

We have $2\frac{1}{2} + \frac{1}{3}$ of $\frac{3}{5} + 5\frac{2}{10} + \frac{3}{8} = 2\frac{1}{2} + \frac{1}{5} + 5\frac{1}{5} + \frac{3}{8}$

$$= 2 + 5 + \frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{3}{8}$$

$$= 7 + \frac{20}{40} + \frac{8}{40} + \frac{8}{40} + \frac{15}{40}.$$

Now $\quad \frac{20}{40} + \frac{8}{40} + \frac{8}{40} + \frac{15}{40}$

$= \quad 20$ times the $\frac{1}{40}$th part of unity

$\quad + \ 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\quad + \ 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\quad + 15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \quad 51 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \frac{51}{40}$;

$\therefore$ sum reqd. $= 7 + \frac{51}{40} = 7 + 1 + \frac{11}{40} = 8 + \frac{11}{40}$.

The process is shortly represented thus :—

$$2\tfrac{1}{2} + \tfrac{1}{3} \text{ of } \tfrac{3}{5} + 5\tfrac{2}{10} + \tfrac{3}{8} = 2\tfrac{1}{2} + \tfrac{1}{5} + 5\tfrac{1}{5} + \tfrac{3}{8} = 2 + 5 + \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{3}{8}$$

$$= 7 + \frac{20 + 8 + 8 + 15}{40} = 7 + \frac{51}{40}$$

$$= 8\tfrac{11}{40}.$$

Ex. X.

1. Add together

(1) $\frac{1}{2}$ and $\frac{1}{4}$. (2) $\frac{2}{3}$ and $\frac{1}{6}$. (3) $\frac{1}{5}$ and $\frac{1}{2}$.

(4) $\frac{2}{3}$ and $\frac{3}{10}$. (5) $\frac{11}{14}$ and $\frac{9}{21}$. (6) $\frac{12}{14}$ and $\frac{10}{21}$.

(7) $1\frac{1}{2}$ and $3\frac{3}{4}$. (8) $2\frac{3}{4}$ and $5\frac{6}{7}$. (9) $8\frac{9}{10}$ and $10\frac{5}{12}$.

(10) $11\frac{12}{13}$ and $\frac{3}{39}$. (11) $\frac{15}{18}$ and $\frac{7}{21}$. (12) $\frac{7}{24}$ and $\frac{5}{27}$.

(13) $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{7}{8}$.

(14) $\frac{11}{12}$, $\frac{13}{15}$, $\frac{12}{18}$, and $\frac{7}{24}$.

(15) $\frac{1}{3}$ of $\frac{3}{5}$, $\frac{2}{7}$ of $\frac{35}{36}$, $\frac{7}{18}$, $\frac{8}{25}$, and $38\frac{9}{10}$.

(16) $3\frac{3}{4}$ of $5\frac{1}{3}$, $\frac{3}{5}$, $\frac{3}{8}$ of $\frac{5}{50}$, and $101\frac{11}{121}$.

(17) $\frac{1}{2}$, $\frac{11}{12}$, $\frac{21}{22}$, $\frac{31}{32}$, $\frac{41}{42}$, and $\frac{51}{52}$.

(18) $\frac{1}{3}$, $\frac{2}{15}$, $\frac{3}{27}$, $\frac{4}{39}$, and $\frac{5}{78}$.

2. Find the value of

(1) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$. (2) $\frac{1}{6} + \frac{1}{7} + \frac{2}{8} + \frac{2}{9} + \frac{2}{10}$.

(3) $\frac{1}{12} + \frac{2}{15} + \frac{3}{18} + \frac{4}{21}$. (4) $\frac{5}{24} + \frac{6}{27} + \frac{7}{30} + \frac{8}{33}$.

(5) $\frac{2}{4} + \frac{2}{8} + \frac{3}{12} + \frac{4}{16}$. (6) $\frac{2}{5} + \frac{2}{10} + \frac{3}{15} + \frac{4}{20}$.

(7) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$.

(8) $1\frac{1}{2} + 2\frac{2}{3} + 3\frac{3}{4} + 4\frac{4}{5} + 5\frac{5}{6} + 6\frac{6}{7} + 7\frac{7}{8} + 8\frac{8}{9} + 9\frac{9}{10}$.

(9) $\frac{1}{12} + \frac{2}{21} + \frac{3}{27} + \frac{4}{72} + \frac{5}{36} + \frac{6}{63}$.

(10) $\frac{4}{5}$ of $\frac{6}{7}$ of $3\frac{19}{24} + \frac{7}{15} + \frac{19}{30}$ of $\frac{3}{76} + 2$.

(11) $2\frac{1}{9}$ of $\frac{3}{11}$ of $2\frac{2}{5} + 6\frac{3}{10} + \frac{13}{14} + 22\frac{1}{2}$.

(12) $\frac{3}{11} + \frac{2}{9}$ of $\frac{4}{22}$ of $\frac{3}{4} + 7\frac{7}{8} + 33\frac{1}{3} + 66\frac{2}{3}$.

(13) $\dfrac{2\frac{1}{2}}{3\frac{1}{3}}$ of $1\frac{1}{3}$ of $\frac{2}{5} + \frac{3}{8} + \frac{7}{10} + 28\frac{3}{11}$.

(14) $\dfrac{3\frac{1}{3}}{8\frac{8}{9}} + \dfrac{5\frac{1}{5}}{10\frac{5}{6}} + \frac{9}{100}$ of $3\frac{1}{3} + 28\frac{2}{35}$.

(15) $\frac{51}{72} + \frac{72}{81} + \frac{81}{102} + \frac{102}{306}$ of $\frac{1}{2} + 82$.

SECTION IV. SUBTRACTION OF FRACTIONS.

100. Rule. Having reduced the fractions as in Addition, subtract the integral part of the subtrahend from that of the minuend.

Reduce the fractional parts of the minuend and the subtrahend to their equivalent fractions having the least common denominator, and subtract the new numerator of the subtrahend, if possible, from that of the minuend, and put down the difference as the numerator of the remainder, and the least common denominator as its denominator. If the new numerator of the minuend is less than that of the subtrahend, take 1 from the integral remainder already found, add it to the fractional part of the minuend, reduced as aforesaid, and having reduced the mixed number so obtained to an improper fraction, from its numerator subtract the new numerator of the subtrahend, and proceed as before.

Reason for the Rule. To subtract one number from another is to subtract separately the integral and the fractional parts of the former from the corresponding parts of the latter; and to subtract one fraction from another is to subtract the number of parts of unity contained in the former from the number of parts in the latter, when they are reduced to a common denominator. When the number of parts of unity contained in the minuend is less than that in the subtrahend, we borrow 1 or an entire unit from the integral remainder, and

add it, or in other words, the number of parts in an entire unit. to the number of parts in the minuend, and subtract from the result the number of parts in the subtrahend.

Ex. 1. Subtract $2\frac{1}{4}$ from $6\frac{1}{3}$.

We have
$$\begin{aligned} 6\tfrac{1}{3} - 2\tfrac{1}{4} &= (6-2) + \tfrac{1}{3} - \tfrac{1}{4} \\ &= 4 + \tfrac{4}{12} - \tfrac{3}{12} \\ &= 4\tfrac{1}{12}. \end{aligned}$$

Ex. 2. Subtract $3\frac{5}{6}$ from $6\frac{1}{4}$.

We have
$$\begin{aligned} 6\tfrac{1}{4} - 3\tfrac{5}{6} &= (6-3) + \tfrac{1}{4} - \tfrac{5}{6} \\ &= 3 + \tfrac{3}{12} - \tfrac{10}{12} \\ &= 2 + (1 + \tfrac{3}{12}) - \tfrac{10}{12} \\ &= 2 + \tfrac{15}{12} - \tfrac{10}{12} \\ &= 2\tfrac{5}{12}. \end{aligned}$$

Ex. 3. Subtract $\frac{3}{4}$ from 3.

We have
$$\begin{aligned} 3 - \tfrac{3}{4} &= 2 + 1 - \tfrac{3}{4} = 2 + \tfrac{4}{4} - \tfrac{3}{4} \\ &= 2\tfrac{1}{4}. \end{aligned}$$

Ex. 4. Find the value of $\frac{3}{4}$ of $\frac{2}{5} - \frac{1}{3} + \frac{7}{8} - \frac{3}{5}$ of $\frac{5}{12}$.

We have
$$\begin{aligned} \tfrac{3}{4} \text{ of } \tfrac{2}{5} - \tfrac{1}{3} + \tfrac{7}{8} - \tfrac{3}{5} \text{ of } \tfrac{5}{12} &= \tfrac{3}{10} - \tfrac{1}{3} + \tfrac{7}{8} - \tfrac{1}{4} \\ &= \tfrac{3}{10} + \tfrac{7}{8} - (\tfrac{1}{3} + \tfrac{1}{4}), \end{aligned}$$

for, $\frac{3}{10}$ and $\frac{7}{8}$ are meant to be added together, and $\frac{1}{3}$ and $\frac{1}{4}$ are both meant to be subtracted from the sum of $\frac{3}{10}$ and $\frac{7}{8}$;

and
$$\begin{aligned} \tfrac{3}{10} + \tfrac{7}{8} - (\tfrac{1}{3} + \tfrac{1}{4}) &= \tfrac{12}{40} + \tfrac{35}{40} - (\tfrac{4}{12} + \tfrac{3}{12}) \\ &= \tfrac{47}{40} - \tfrac{7}{12} \\ &= \tfrac{141}{120} - \tfrac{70}{120} \\ &= \tfrac{71}{120}. \end{aligned}$$

Ex. 5. What proper fraction must be added to $2\frac{3}{4}$ to make the result an integer?

Evidently, the required fraction must be such, that being added to $\frac{3}{4}$, it will give 1 for the sum;

$\therefore$ the fraction required $= 1 - \frac{3}{4} = \frac{4}{4} - \frac{3}{4} = \frac{1}{4}$.

Ex. XI.

1. Find the difference between

(1) $\frac{1}{2}$ and $\frac{2}{3}$. (2) $\frac{2}{3}$ and $\frac{3}{5}$. (3) $\frac{3}{4}$ and $\frac{2}{5}$.
(4) $\frac{2}{3}$ and $\frac{3}{4}$. (5) $\frac{3}{4}$ and $\frac{4}{5}$. (6) $\frac{4}{5}$ and $\frac{5}{6}$.
(7) $\frac{3}{4}$ and $\frac{3}{5}$. (8) $\frac{2}{3}$ and $\frac{7}{10}$. (9) $\frac{3}{16}$ and $\frac{2}{12}$.
(10) $2\frac{3}{4}$ and $7\frac{2}{8}$. (11) $5\frac{3}{4}$ and $2\frac{2}{3}$. (12) $12\frac{5}{12}$ and $17\frac{3}{16}$.
(13) $22\frac{2}{9}$ and $11\frac{2}{5}$ of $\frac{3}{19}$. (14) 25 and $\frac{3}{5}$ of $\frac{5}{9}$.
(15) $\frac{2}{3}$ of $\frac{3}{11}$ and $17\frac{7}{33}$. (16) $\frac{2}{7}$ of $\frac{56}{58}$ and $\frac{3}{29}$ of $\frac{9}{12}$.
(17) $11\frac{22}{222}$ and $3\frac{3}{37}$. (18) $1\frac{2}{20}$ and $\frac{2}{3}$ of $\frac{3}{10}$ of $\frac{5}{16}$.
(19) 2 and $\frac{2}{3}$. (20) 22 and $33\frac{2}{3}$.

2. Find the value of

(1) $\frac{1}{2}+\frac{2}{3}-\frac{1}{4}-\frac{2}{5}$ (2) $\frac{2}{3}-\frac{2}{5}+\frac{2}{7}-\frac{2}{9}$.
(3) $\frac{3}{5}+\frac{7}{10}-\frac{2}{15}-\frac{2}{25}$. (4) $\frac{2}{9}$ of $\frac{3}{4}-\frac{2}{15}+\frac{3}{8}$.
(5) $\frac{2}{21}+\frac{2}{22}-\frac{2}{11}+\frac{2}{12}$. (6) $\frac{99}{100}-\frac{9}{10}+\frac{4}{30}-\frac{40}{330}$.

3. What fraction must be added to $\frac{3}{4}$ to make the sum equal to the difference between $2\frac{1}{2}$ and $1\frac{1}{4}$?

4. What proper fraction must be added to $5\frac{7}{15}$ to make the result an integer?

5. What fraction must be subtracted from $\frac{8}{15}$ to give $\frac{1}{2}$?

6. What fraction must be added to $\frac{7}{24}$ to give $\frac{2}{3}$?

SECTION V. MULTIPLICATION OF FRACTIONS.

101. Before giving the Rule for the Multiplication of fractions, let us see how far the meaning we have attached to the operation of Multiplication in our Definition in Art. 13, applies to the case in which the multiplier is not an integer.

When a number is taken to the extent of its *double*, *triple*, &c., we say that it is taken 2, 3, &c. times, *i. e.*, is multiplied by 2, 3, &c. Similarly, when a number, whether integral or fractional, is taken to the extent of its *half*, *one-third*, *two-thirds*, *&c.*, we may, by a *stretch of language*, say, that it is taken *half*, *one-third*, *two-thirds*, &c. times, or rather parts of a time, *i. e.*, by our Definition in Art. 13, that it is multiplied by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, &c.

Thus, to multiply any number $2\frac{1}{3}$ by $\frac{2}{7}$ is to take $\frac{2}{7}$ of $2\frac{1}{3}$, *i. e.*, $\frac{2}{7}$ of $\frac{7}{3}$; *i. e.*, it is to divide $\frac{7}{3}$ into 7 equal parts, and then to take 2 of such parts;

$$\therefore 2\tfrac{1}{3} \times \tfrac{2}{7} = \left(\frac{7}{3} \div 7\right) \times 2 = \frac{7}{3 \times 7} \times 2 = \frac{7 \times 2}{3 \times 7} \text{ (Art 89, Prop. II)}$$

$$= \tfrac{2}{3} \text{ (after striking out 7).}$$

Similarly, if there is another factor $\frac{3}{4}$, we have

$$\tfrac{7}{3} \times \tfrac{2}{7} \times \tfrac{3}{4} = \frac{7 \times 2}{3 \times 7} \times \frac{3}{4} = \frac{7 \times 2 \times 3}{3 \times 7 \times 4} = \tfrac{1}{2},$$

(after canceling the common factors 7, 3, and 2).

Hence we deduce the Rule for the Multiplication of fractions given in the next Article.

102. Rule. Reduce the factors, if necessary, to the form of simple fractions, multiply all the numerators together for the numerator of product, and all the denominators together for its denominator, and simplify the result by canceling all the factors common to the numerator and the denominator.

Ex. Multiply $\frac{2}{3}$, $\frac{4}{5}$, $\frac{3}{4}$ and $\frac{5}{6}$ together.

By the Rule we have

$$\tfrac{2}{3} \times \tfrac{4}{5} \times \tfrac{3}{4} \times \tfrac{5}{6} = \frac{2 \times 4 \times 3 \times 5}{3 \times 5 \times 4 \times 6} = \tfrac{2}{6} = \tfrac{1}{3}.$$

103. In the Multiplication of fractions, the order of the factors is immaterial.

$$\text{Thus } \tfrac{2}{3} \times \tfrac{6}{7} = \frac{2 \times 6}{3 \times 7} = \frac{6 \times 2}{7 \times 3} \text{ (Art. 41)}$$

$$= \tfrac{6}{7} \times \tfrac{2}{3}.$$

104. It may be here observed that a compound fraction denotes the product of the component fractions.

Ex. XII.

1. Multiply—

(1) $\frac{1}{2}$ by $\frac{2}{3}$. (2) $\frac{2}{3}$ by $\frac{3}{4}$. (3) $1\frac{1}{4}$ by $\frac{4}{5}$.

(4) $\frac{5}{6}$ by $\frac{3}{10}$. (5) $\frac{6}{7}$ by $\frac{5}{12}$. (6) $\frac{7}{8}$ by $\frac{20}{21}$.

(7) $\frac{1}{2}$ of $\frac{3}{4}$ by $\frac{4}{5}$ of $\frac{5}{7}$. (8) $3\frac{2}{3}$ of $\frac{3}{22}$ by $\frac{15}{16}$.

(9) $\frac{1}{2}$ of $\frac{3}{4}$ by $\frac{4}{5}$ of $\frac{6}{7}$. (10) $\frac{9}{10}$ of $\frac{20}{11}$ by $\frac{22}{27}$.

(11) $\frac{9}{20}$ of $\frac{6}{7}$ by $\frac{30}{25}$ of $\frac{28}{24}$. (12) $\frac{13}{57}$ of $\frac{27}{32}$ by $\frac{8}{39}$ of $1\frac{7}{12}$.

2. Find the continued product of

(1) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{4}$, and $\frac{3}{5}$.

(2) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, and $\frac{8}{9}$.

(3) $\frac{1}{2}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$, $\frac{5}{11}$, and $\frac{6}{14}$.

(4) $\frac{1}{2}$, $\frac{8}{17}$, $\frac{23}{24}$, $\frac{34}{35}$, and $\frac{45}{46}$.

(5) $\frac{15}{16}$, $\frac{35}{36}$, $1\frac{8}{55}$, $1\frac{2}{75}$, and $\frac{5}{7}$.

(6) $\frac{7}{23}$, $\frac{19}{21}$, $\frac{92}{100}$, $\frac{5}{57}$, and $\frac{9}{10}$.

3. Find the value of

(1) $\frac{1}{2}$ of $\frac{8}{9} \times \frac{9}{10} + 3\frac{2}{3} - 2\frac{3}{5}$.

(2) $\frac{22}{21} + \frac{12}{13} \times \frac{117}{120} \times \frac{5}{9} - \frac{1}{105}$.

(3) $3\frac{2}{3} - \frac{2}{5} \times \frac{5}{8} + \frac{3}{8} \times \frac{9}{22}$.

SECTION VI. DIVISION OF FRACTIONS.

105. The meaning which we have attached to Division in Art. 14, is applicable to the Division of an integer or a fraction by a fraction, as we have already seen in Art. 93, and as will further appear from the following Examples.

Ex. 1. Divide 3 by $\frac{4}{5}$.

Here we have to find how often the secondary units in $\frac{4}{5}$ are contained in 3 primary units, *i. e.*, how often 4 secondary units, whereof each is $\frac{1}{5}$th of 1, are contained in 3×5 secondary units of the same value;

$$\therefore \text{ the quotient reqd.} = 3 \times 5 \div 4 = \frac{3 \times 5}{4}$$

$$= \frac{\text{dividend} \times \text{denr. of divisor}}{\text{numr. of divisor}}.$$

Ex. 2. Divide $\frac{2}{3}$ by $\frac{3}{5}$.

Here we have

quotient reqd. $= \frac{2}{3} \div \frac{3}{5}$

$$= \frac{2 \times 5}{3 \times 5} \div \frac{3 \times 3}{5 \times 3} \text{ (Art.89, Prop.I)}$$

$$= \frac{2 \times 5}{15} \div \frac{3 \times 3}{15}$$

$= 2 \times 5$ secondary units whereof each is $\frac{1}{15}$ of 1 $\div 3 \times 3$...

$$= \frac{2 \times 5}{3 \times 3}$$

$$= \frac{\text{numr. of dividend} \times \text{denr. of divisor}}{\text{denr. of dividend} \times \text{numr. of divisor}}.$$

Hence we deduce the Rule for the Division of fractions given in the following Article.

106. Rule. Reduce the dividend and the divisor, if necessary, to the form of simple fractions, invert the divisor, and then proceed as in Multiplication.

Ex. Divide $2\frac{1}{2}$ of $\frac{3}{4}$ by $\frac{3}{8}$ of $\frac{2}{3}$.

By the Rule we have $(2\frac{1}{2} \text{ of } \frac{3}{4}) \div (\frac{3}{8} \text{ of } \frac{2}{3}) = (\frac{5}{2} \text{ of } \frac{3}{4}) \div (\frac{3}{8} \text{ of } \frac{2}{3})$

$$= \frac{5 \times 3}{2 \times 4} \div \frac{3 \times 2}{8 \times 3}$$

$$= \frac{5 \times 3}{2 \times 4} \times \frac{8 \times 3}{3 \times 2}$$

$$= \frac{5 \times 3 \times 8 \times 3}{2 \times 4 \times 3 \times 2} = \frac{15}{2}$$

$$= 7\frac{1}{2}$$

107. It now remains to be seen how far the second meaning of Division noticed in Art. 50, is applicable to the division of a number by a fraction. Division in this sense is the method of finding the magnitude of each part of the dividend when it is divided into the number of parts indicated by the divisor, or in other words, of finding a part of the dividend, such that when multiplied by the divisor, it will reproduce the dividend. In this sense, to divide $\frac{2}{3}$ by $\frac{3}{5}$ is to find a fraction, such that when multiplied by $\frac{3}{5}$ in the sense in which Multiplication is understood in Art. 101, it will produce $\frac{2}{3}$.

Now the quotient of $\frac{2}{3} \div \frac{3}{5}$ according to the Rule in Art. 106 is $\frac{2 \times 5}{3 \times 3}$,

and $\frac{2 \times 5}{3 \times 3} \times \frac{3}{5} = \frac{2}{3}$.

Hence, the quotient obtained by the preceding Rule also expresses numerically the magnitude of a part of the dividend, such that when multiplied by the divisor, it will reproduce the dividend.

108. From the above we see that in the Division of fractions also,

quotient $\times$ divisor $=$ dividend.

Ex. XIII.

1. Divide—

(1) 1 by $\frac{1}{2}$. (2) 2 by $\frac{2}{3}$. (3) $1\frac{1}{2}$ by $2\frac{2}{3}$.
(4) $\frac{5}{16}$ by $\frac{5}{6}$. (5) $\frac{9}{10}$ by $\frac{9}{90}$. (6) $3\frac{4}{30}$ by $4\frac{7}{10}$.
(7) $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ by $\frac{5}{12}$ of $\frac{2}{15}$ of $\frac{7}{14}$.
(8) $\frac{1}{5}$ of $\frac{1}{6}$ of $\frac{1}{7}$ by $\frac{1}{8}$ of $\frac{1}{9}$ of $\frac{1}{10}$.
(9) $1\frac{1}{4}$ of $2\frac{1}{5}$ of $3\frac{1}{12}$ by $2\frac{1}{2}$ of $3\frac{2}{5}$ of $4\frac{1}{9}$.
(10) $7\frac{1}{3}$ of $\frac{3}{11}$ of $\frac{12}{91}$ by $\frac{7}{13}$ of $\frac{5}{14}$ of $\frac{9}{15}$.
(11) $101\frac{1}{3}$ of $\frac{3}{5}$ of $\frac{9}{13}$ by $\frac{19}{32}$ of $\frac{64}{65}$.
(12) $1\frac{1}{99}$ of $2\frac{2}{9}$ of $3\frac{3}{9}$ by $3\frac{7}{11}$ of $4\frac{5}{13}$ of $\frac{1}{3}$.

2. What number multiplied by $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ will give $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$?

3. What number must be multiplied by the continued product of the reciprocals of the first five integers to give the reciprocal of the sixth?

4. Divide the difference between ten and one-tenth by the sum of one and one-tenth.

MISCELLANEOUS QUESTIONS AND EXAMPLES.

109. In simplifying expressions containing fractions, the Rule in Art. 76 should be borne in mind.

Ex. 1. Simplify $\dfrac{2\frac{1}{2} \text{ of } \frac{3}{10} - \frac{1}{3} \text{ of } \frac{1}{4}}{2\frac{1}{2}} - (\frac{1}{10} - \frac{1}{100}) + \dfrac{2\frac{3}{4}}{2\frac{1}{5}} \div 3\frac{1}{3}$.

The given expression

$$= \frac{\frac{5}{2} \times \frac{3}{10} - \frac{1}{3} \times \frac{1}{4}}{\frac{5}{2}} - \frac{10-1}{100} + \frac{\frac{11}{4}}{\frac{11}{5}} \div \frac{10}{3}$$

$$= \frac{\frac{3}{4} - \frac{1}{12}}{\frac{5}{2}} - \frac{9}{100} + \frac{11 \times 5}{4 \times 11} \times \frac{3}{10} = \frac{\frac{9}{12} - \frac{1}{12}}{\frac{5}{2}} - \frac{9}{100} + \frac{3}{8}$$

$$= \frac{\frac{8}{12}}{\frac{5}{2}} - \frac{9}{100} + \frac{3}{8} = \frac{8 \times 2}{12 \times 5} - \frac{9}{100} + \frac{3}{8}$$

$$= \frac{4}{15} - \frac{9}{100} + \frac{3}{8} = \frac{160}{600} - \frac{54}{600} + \frac{225}{600}$$

$$= \frac{331}{600}.$$

Ex. 2. *A* owns $\frac{2}{5}$ of an estate and *B* $\frac{3}{10}$ of the same. For a certain price, *A* offers to sell you $\frac{1}{4}$ of his share, and for the same price, *B* offers to sell $\frac{2}{5}$ of his. Which is the more profitable offer to you; what is your gain by accepting it; and what share of his property must *B* offer to make the two offers equally profitable?

Here there are three questions. Take them one after another.

(1) To find which is the more profitable offer.

For the same price,

A offers $\frac{1}{4}$ of his share, *i. e.*, $\frac{1}{4}$ of $\frac{2}{5}$ of the estate,
and *B* ... $\frac{2}{5}$ ... ... $\frac{2}{5}$ of $\frac{3}{10}$ ...

Hence, the question is reduced to the comparison of the two fractions $\frac{1}{4}$ of $\frac{2}{5}$ and $\frac{2}{5}$ of $\frac{3}{10}$

i. e., $\frac{1}{10}$ and $\frac{3}{25}$

i. e., $\frac{5}{50}$ and $\frac{6}{50}$.

The latter, being the greater, *B*'s offer is the more profitable of the two.

(2) To find the gain.

This is the difference between the two offers, *i. e.*, between $\frac{5}{50}$ and $\frac{6}{50}$ and equals $\frac{6}{50} - \frac{5}{50} = \frac{1}{50}$.

So that by accepting *B's* offer, the purchaser gets $\frac{1}{50}$ of the estate more than what he gets by accepting the other offer.

(3) To find the share of his property that *B* must agree to sell, to make his offer equal to *A*'s.

A's offer is $\frac{1}{10}$ of the whole estate,
and *B* owns $\frac{3}{10}$ ... ...

Hence the question is reduced to finding a fraction such that $\frac{3}{10}$ being multiplied by it, will produce $\frac{1}{10}$.

This fraction $= \frac{1}{10} \div \frac{3}{10} = \frac{1}{3}$.

So that *B* must agree to sell $\frac{1}{3}$ of his property to make his offer equal to *A*'s.

Ex. 3. What is the total number of rupees whereof $\frac{1}{2}$ being spent for one purpose, and $\frac{1}{5}$ for another, there remain 6 rupees left?

Taking $\frac{1}{2} + \frac{1}{5}$ from the whole or unity,

we have $1 - (\frac{1}{2} + \frac{1}{5}) = 1 - \frac{7}{10} = \frac{3}{10}$.

Now $\frac{3}{10} \times$ the total no. reqd. $= 6$;

$$\therefore \quad \text{the total no. reqd.} = 6 \div \frac{3}{10} = \frac{6 \times 10}{3} = 20.$$

Ex. 4. A book contains 4 chapters, whereof the 1st and the 2nd together contain $\frac{13}{20}$ of the whole number of pages in the book; the 2nd and the 3rd together, $\frac{3}{5}$ of the whole number; and the 2nd contains twice as many pages as the 1st and the 3rd together: and there are 20 pages in the 4th chapter. What is the total number of pages in the book, and how many pages are there in each of the first 3 chapters?

Here, we have

no. of pages in Ch. I + no. in Ch. II $= \frac{13}{20}$ of total no.;
... ... Ch. II + ... Ch. III $= \frac{3}{5}$ ... ...
$\therefore$ ... ... Ch. I + ... Ch. III + 2 × no. in Ch. II
$= (\frac{13}{20} + \frac{3}{5})$ of total no. $= \frac{25}{20}$ of total no..

But by the question,

no. in Ch. II $= 2 \times$ (no. in Ch. I + no. in Ch. III);

$\therefore$ no. in Ch. I + no. in Ch. III + $2 \times 2 \times$ (no. in Ch. I + no. in Ch. III)

or $5 \times$ (no. in Ch. I + no. in Ch. III) $= \frac{25}{20}$ of total no.,

and $\therefore$ no. in Ch. I + no. in Ch. III $= \frac{1}{5}$ of $\frac{25}{20}$ ...

$= \frac{1}{4}$ ... .

Hence no. in Ch. II $= 2 \times \frac{1}{4}$ ...

$= \frac{1}{2}$ ... ,

and $\therefore$ no. in Ch. I + $\frac{1}{2}$ of total no. $= \frac{13}{20}$ ... ,

or no. in Ch. I $= \frac{13}{20} - \frac{1}{2}$ ...

$= \frac{3}{20}$ ... .

Similarly, no. in Ch. III $= \frac{3}{5} - \frac{1}{2}$ ...

$= \frac{1}{10}$ ... .

Hence no. of pages in the first 3 Ch. $= \frac{1}{2} + \frac{3}{20} + \frac{1}{10}$ ...

$= \frac{15}{20}$ ... ,

and ... ... Ch. IV $= 1 - \frac{15}{20}$ ...

$= \frac{5}{20}$ ... .

But by the question,

no. of pages in Ch. IV $= 20$;

$\therefore \frac{5}{20}$ of total no. $= 20$;

and $\therefore$ the total no. of pages $= 20 \div \frac{5}{20}$

$= 80.$

And hence no. of pages in Ch. I $= \frac{3}{20}$ of $80 = 12$,

......................Ch. II $= \frac{1}{2}$ of $80 = 40$,

and......................Ch. III $= \frac{1}{10}$ of $80 = 8$.

Ex. 5. A man dies leaving his father who gets $\frac{1}{6}$ of his estate, 3 widows who divide $\frac{1}{8}$ of his estate equally amongst themselves, and 2 sons and 3 daughters who take the remainder in such a manner that each son gets twice as much as each daughter. Divide the estate into the least number of parts such that each claimant may get an integral number of those parts.

The father gets $\frac{1}{6}$ of the estate;

each widow, $\frac{1}{3}$ of $\frac{1}{8}$ or $\frac{1}{24}$..............;

and there remain $1 - (\frac{1}{6} + \frac{1}{8})$ or $\frac{17}{24}$...............

Now each son takes 2 parts while each daughter takes 1; $\therefore$ for the children there must be $2 + 2 + 1 + 1 + 1$ or 7 parts, whereof each son takes 2, and each daughter 1.

This is the difference between the two offers, *i. e.*, between $\frac{5}{50}$ and $\frac{6}{50}$ and equals $\frac{6}{50} - \frac{5}{50} = \frac{1}{50}$.

So that by accepting *B's* offer, the purchaser gets $\frac{1}{50}$ of the estate more than what he gets by accepting the other offer.

(3) To find the share of his property that *B* must agree to sell, to make his offer equal to *A*'s.

A's offer is $\frac{1}{10}$ of the whole estate,
and *B* owns $\frac{3}{10}$ ... ...

Hence the question is reduced to finding a fraction such that $\frac{3}{10}$ being multiplied by it, will produce $\frac{1}{10}$.

This fraction $= \frac{1}{10} \div \frac{3}{10} = \frac{1}{3}$.

So that *B* must agree to sell $\frac{1}{3}$ of his property to make his offer equal to *A*'s.

Ex. 3. What is the total number of rupees whereof $\frac{1}{2}$ being spent for one purpose, and $\frac{1}{5}$ for another, there remain 6 rupees left?

Taking $\frac{1}{2} + \frac{1}{5}$ from the whole or unity,

we have $1 - (\frac{1}{2} + \frac{1}{5}) = 1 - \frac{7}{10} = \frac{3}{10}$.

Now $\frac{3}{10} \times$ the total no. reqd. $= 6$;

$$\therefore \quad \text{the total no. reqd.} = 6 \div \frac{3}{10} = \frac{6 \times 10}{3} = 20.$$

Ex. 4. A book contains 4 chapters, whereof the 1st and the 2nd together contain $\frac{13}{20}$ of the whole number of pages in the book; the 2nd and the 3rd together, $\frac{3}{5}$ of the whole number; and the 2nd contains twice as many pages as the 1st and the 3rd together: and there are 20 pages in the 4th chapter. What is the total number of pages in the book, and how many pages are there in each of the first 3 chapters?

Here, we have

no. of pages in Ch. I + no. in Ch. II $= \frac{13}{20}$ of total no.;
... ... Ch. II + ... Ch. III $= \frac{3}{5}$ ... ...
$\therefore$ ... ... Ch. I + ... Ch. III + 2 × no. in Ch. II
$= (\frac{13}{20} + \frac{3}{5})$ of total no. $= \frac{25}{20}$ of total no.

But by the question,

no. in Ch. II $= 2 \times$ (no. in Ch. I + no. in Ch. III);

$\therefore$ no. in Ch. I + no. in Ch. III + $2 \times 2 \times$ (no. in Ch. I + no. in Ch. III)

or $5 \times$ (no. in Ch. I + no. in Ch. III) $= \frac{25}{20}$ of total no.;

and $\therefore$ no. in Ch. I + no. in Ch. III $= \frac{1}{5}$ of $\frac{25}{20}$ ...

$= \frac{1}{4}$ ... .

Hence no. in Ch. II $= 2 \times \frac{1}{4}$ ...

$= \frac{1}{2}$ ... ,

and $\therefore$ no. in Ch. I + $\frac{1}{2}$ of total no. $= \frac{13}{20}$ ... ,

or no. in Ch. I $= \frac{13}{20} - \frac{1}{2}$ ...

$= \frac{3}{20}$ ... .

Similarly, no. in Ch. III $= \frac{3}{5} - \frac{1}{2}$ ...

$= \frac{1}{10}$ ... .

Hence no. of pages in the first 3 Ch. $= \frac{1}{2} + \frac{3}{20} + \frac{1}{10}$ ...

$= \frac{15}{20}$ ... ;

and ... ... Ch. IV $= 1 - \frac{15}{20}$ ...

$= \frac{5}{20}$ ... .

But by the question,

no. of pages in Ch. IV $= 20$;

$\therefore \frac{5}{20}$ of total no. $= 20$;

and $\therefore$ the total no. of pages $= 20 \div \frac{5}{20}$

$= 80.$

And hence no. of pages in Ch. I $= \frac{3}{20}$ of $80 = 12$,

....................Ch. II $= \frac{1}{2}$ of $80 = 40$,

and....................Ch. III $= \frac{1}{10}$ of $80 = 8$.

Ex. 5. A man dies leaving his father who gets $\frac{1}{6}$ of his estate, 3 widows who divide $\frac{1}{8}$ of his estate equally amongst themselves, and 2 sons and 3 daughters who take the remainder in such a manner that each son gets twice as much as each daughter. Divide the estate into the least number of parts such that each claimant may get an integral number of those parts.

The father gets $\frac{1}{6}$ of the estate;

each widow, $\frac{1}{3}$ of $\frac{1}{8}$ or $\frac{1}{24}$.............. ;

and there remain $1 - (\frac{1}{6} + \frac{1}{8})$ or $\frac{17}{24}$...............

Now each son takes 2 parts while each daughter takes 1; $\therefore$ for the children there must be $2 + 2 + 1 + 1 + 1$ or 7 parts, whereof each son takes 2, and each daughter 1.

Hence each son takes $\frac{2}{7}$ of $\frac{17}{24}$ or $\frac{17}{84}$ of the estate,

and each daughter $\frac{1}{7}$ of $\frac{17}{24}$ or $\frac{17}{168}$ ....

Thus the shares of the claimants are

$\frac{1}{6}$, $\frac{1}{24}$, $\frac{17}{84}$, and $\frac{17}{168}$;

and these fractions reduced to their equivalent ones having the least common denominator, are

$\frac{28}{168}$, $\frac{7}{168}$, $\frac{34}{168}$, and $\frac{17}{168}$.

Hence if we divide the estate into 168 equal parts, the father will get 28, each widow, 7, each son, 34, and each daughter, 17 parts; so that 168 is the number required.

Ex. XIV.

I.

1. What is a Fraction and why is it so called? What are the two systems of fractions in common use?

2. State the method of Notation of fractions. Point out clearly the relation between the value of a fraction and the result of the division of its numerator by its denominator.

3. Shew that the value of a fraction is not altered if its numerator and its denominator are both multiplied or both divided by the same number.

Reduce $\frac{738}{819}$ to its lowest terms.

4. What is a Compound Fraction? Shew how to reduce a Compound Fraction to the form of a Simple Fraction.

Simplify $\frac{3}{100}$ of $\frac{20}{11}$ of $\frac{99}{190}$ of $(\frac{2}{3}+\frac{3}{5})$.

5. What is a Complex Fraction? Shew how to simplify it.

Simplify $\dfrac{1+\frac{2}{3}\text{ of }\frac{3}{4}}{4\frac{1}{5}+\frac{3}{4}}$ of $3\frac{3}{10}$.

6. A workman is offered 6 rupees per week (*i. e.* 7 days) by one employer, and 25 rupees per month (*i. e.* 30 days) by another. Which is the better remuneration?

II.

1. A person inherits $\frac{2}{7}$ of $\frac{3}{5}$ of an estate; he next obtains by gift $\frac{3}{5}$ of that estate; and he lastly purchases $\frac{2}{35}$ of the same. How much of the estate does he now own, and what fraction of the estate remains to be acquired by him, to make him the sole proprietor of the whole?

2. A post has $\frac{1}{5}$ of its length in the mud, $\frac{2}{5}$ in the water, and 12 feet above the water. What is the whole length of the post?

3. A gentleman has two sons and a daughter. His age equals the sum of the ages of his children; the age of the eldest child is $\frac{4}{9}$ of his age; that of the second, $\frac{1}{3}$ of his age; and that of the third, 10 years. Find the ages of the father and the first two children.

4. Two numbers are respectively $\frac{1}{5}$ and $\frac{1}{6}$ of a third; and the difference between this last and the sum of the other two is 19. Find the numbers.

5. Find the difference between the reciprocal of the sum of the first four natural numbers and the sum of the reciprocals of first four even numbers.

6. Find the least fraction which being added to the sum of $\frac{2}{3}$ and $\frac{3}{4}$ will make the result an integer.

III.

1. Find the number of which the double and the triple together exceed the half by $\frac{3}{8}$.

2. If I pay away $\frac{2}{3}$ of my money, then $\frac{3}{4}$ of what remains, and then $\frac{3}{8}$ of what still remains, what fraction of the original amount have I still left?

3. What number must be added to $\dfrac{2+3}{12+13}+\dfrac{3+4}{7+14}$ to make the result equal to $\dfrac{1+2}{2+3}+\dfrac{2+3}{4+6}$?

4. Each of 3 bags contains a certain number of rupees, such that the amount in the first and the third taken together is twice the amount in the second which is 600 rupees; and the amount in the first and the second together

is $\frac{5}{9}$ of the total amount : find the total number of rupees, and the number in the first bag.

5. What number must be taken from $\dfrac{3+3}{4+4}$ to make the result equal to $\dfrac{1+2}{2+3}$?

6. Three boys A, B, and C, have each a certain number of marbles. The number belonging to B is $\frac{1}{3}$ of the number belonging to all the three ; the number belonging to A and B together is $\frac{5}{9}$ of the total number ; and the number belonging to C is 16. Find the total number of marbles, and the number belonging to A.

IV.

1. What do you understand by the Multiplication of a number by a proper fraction?

Give the reason for the Rule for the Multiplication of fractions.

2. Multiply the sum of $\frac{2}{3}$, $\frac{2}{5}$, and $\frac{2}{9}$ by the difference between $\frac{2}{9}$ and $\frac{2}{29}$.

3. What number multiplied by $\frac{3}{5}$ of $\frac{2}{9} - \frac{2}{3}$ of $\frac{1}{10}$ will produce $\frac{3}{7}$ of $\frac{15}{45}$?

4. The sum of the ages of two boys is 24 years, and the difference of their ages is $\frac{2}{5}$ of the age of the younger. What is the age of each?

5. The sum of two fractions is $2\frac{3}{4}$ times their difference, and the greater is $\frac{3}{4}$. Find the less.

6. Find the continued product of $\frac{3}{2}$, $4\frac{2}{3}$, $5\frac{4}{7}$ and $6\frac{2}{13}$.

V.

1. What is the meaning of the quotient arising from the Division of one fraction by another?

State the Rule for the Division of fractions, and give the reason for that Rule.

2. A number divided by 2 becomes half of what it is, and a number divided by $\frac{1}{2}$ becomes double of what it is. Explain clearly the reason for this. What do you understand by Division in the latter case?

3. What number divided by $3\frac{1}{2}$ will produce $5\frac{5}{7}$?

4. What number multiplied by the sum of $\frac{2}{7}$ and $\frac{1}{10}$ will equal the quotient arising from the division of the difference of $\frac{2}{5}$ and $\frac{3}{6}$ by the sum of $\frac{2}{6}$ and $\frac{3}{8}$?

5. Divide the product of $\dfrac{3\frac{3}{4}}{2\frac{1}{3}}$ and $\frac{7}{23}$ by the quotient arising from the division of $\dfrac{4\frac{3}{5}}{3\frac{2}{3}}$ by $2\frac{6}{20}$.

6. By what number must you divide $9\frac{1}{5}$ to make the quotient equal to 23?

VI.

1. Simplify :—

(1) $\dfrac{2 \div \frac{3}{4}}{3 \times \frac{4}{5}} \times \dfrac{5\frac{2}{3}}{7\frac{2}{7}} \div 7.$ (2) $\dfrac{\frac{2}{3} + \frac{3}{4}}{\frac{2}{5} + \frac{4}{7}}$ of $\frac{4}{5}$.

2. *A* owns $\frac{3}{5}$ of an estate, and sells $\frac{2}{5}$ of his share to *B*. How much more of his share must *A* sell to *B* to make their shares equal, and what share of the entire estate will each own in that case?

3. In a certain field there are 66 trees arranged in 3 rows. The number of trees in the 1st row is 3 times that in the 3rd, and 2 times that in the 2nd. How many trees are there in each row?

4. Find the sum of the sum and the difference of $2\frac{1}{2}$ and $3\frac{3}{4}$, without performing the operations of Addition and Subtraction of fractions.

5. What part of $\frac{3}{4}$ is $\frac{2}{3}$, and what part of $\frac{9}{10}$ is $\frac{3}{5}$?

6. A man dies leaving 2 widows, 3 sons, and 4 daughters. His widows are entitled to $\frac{1}{8}$ of his property, to be divided equally between them, and his children to the remainder, to be divided amongst them in such a manner that the share of a son shall be double of that of a daughter. Divide the property into the least number of parts such that each claimant may get an integral number of those parts.

DIVISION II. DECIMALS.

SECTION VII. NOTATION AND NUMERATION OF DECIMALS.

110. We have seen in Art. 79, that decimal fractions are those that consist of secondary units or parts which are tenths, hundredths, &c., of the primary unit. Hence, a decimal is a fraction having 10, 100 *i. e.* 10^2, or some other power of 10, for its denominator, and the number of tenths, hundredths, or other parts that it consists of, for its numerator; and it may be expressed in the same manner as a vulgar fraction.

Thus,

three-tenths, seven hundredths, twenty-six thousandths,

may be written

$$\frac{3}{10}, \qquad \frac{7}{100}, \qquad \frac{26}{1000}.$$

111. The numerator of a decimal fraction may be analyzed into its constituent digits which would represent the number of tenths, the number of hundredths, &c., that compose the fraction.

Thus, take as an example the decimal fraction $\frac{236}{1000}$.

Then
$$\frac{236}{1000} = \frac{200+30+6}{1000} = \frac{200}{1000} + \frac{30}{1000} + \frac{6}{1000}$$
$$= \frac{2}{10} + \frac{3}{100} + \frac{6}{1000}.$$

Take another decimal fraction $\frac{705}{10000}$.

Then
$$\frac{705}{10000} = \frac{700+0\times10+5}{10000} = \frac{700}{10000} + \frac{0\times10}{10000} + \frac{5}{10000}$$
$$= \frac{7}{100} + \frac{0}{1000} + \frac{5}{10000}.$$

From the above we see that a decimal fraction may be regarded as a number composed of the figures of its numerator, the first figure on the left representing so many tenths or hundredths, or the like, as the case may be, and the others having their local values decreasing tenfold at each step towards the right.

112. The preceding Article *suggests* another method of Notation for decimals, as will be seen below.

In the Common System of Notation, the local value of every figure *increases* tenfold at each step towards the *left*, *i. e.*, *decreases* tenfold at each step towards the *right*. If then we consider this scale of tenfold decrease of local value *extended* to the right of the units' place, and put figures there, separated from the other figures by a mark such as a dot, we shall have to the right of the dot a series of figures whose local values are so many *tenths*, *hundredths*, &c., and these figures will in every case represent some decimal fraction or other.

Thus, $$267{\cdot}203 = 267 + \frac{2}{10} + \frac{0}{100} + \frac{3}{1000}$$
$$= 267\frac{203}{1000},$$

i. e., 203 represents the decimal fraction $\frac{203}{1000}$.

So, $$\cdot023 = \frac{0}{10} + \frac{2}{100} + \frac{3}{1000}$$
$$= \frac{23}{1000},$$

i. e., $\cdot023$ represents the decimal fraction $\frac{23}{1000}$.

And so in other cases.

Hence we may deduce the following Rule for the Notation of decimals :—

RULE. Having written in figures the integral part of the number, if any, place a dot on its right, and on the right of the dot, write the numerator of the decimal fraction, preceded by ciphers, if necessary, to make the number of figures to the right of the dot equal to the number of ciphers in the denominator.

Ex. Express in figures, two hundred and fifty-six, and fifty-three thousandths.

Here, there being 3 ciphers in the denominator, the numerator 53 must have 1 cipher prefixed to it, before it is put after the dot; and the number will be written thus :—

$$256{\cdot}053.$$

And hence a decimal expressed in the above mode can be reduced to the form of a vulgar fraction by the following Rule:—

RULE. Write the decimal, omitting the dot and the ciphers just after it, as the numerator, and 1 followed by as many ciphers as there are figures to the right of the dot, for the denominator.

Ex. Express ·028 as a vulgar fraction.

We have $\cdot 028 = \frac{28}{1000} = \frac{7}{250}$.

DEF. The dot separating the integral part of a number from the decimal is called the DECIMAL POINT, and the places of figures to the right of the dot are called the DECIMAL PLACES.

113. The above is the usual method of Notation for decimals. We have thus a uniform *ascending* and *descending* scale of Notation extending without limit to the *left* and the *right* of the units' place, the former or the ascending part of the scale consisting of tens, hundreds, &c., and being sufficient for the expression of all possible integers; and the latter or the descending part of the scale consisting of tenths, hundredths, &c., and being sufficient for the expression of all possible *decimal* fractions.

It is this capability of being expressed in a uniform system of Notation with integers, that constitutes the peculiar advantage of decimals over vulgar fractions, and makes them peculiarly adapted for numerical calculation, as the student will hereafter see.

114. The above system is evidently convenient and complete for the expression of *decimal* fractions. It now remains to be seen whether *every possible* fraction can be expressed in this system.

Take the fraction $\frac{1}{2}$.

Then $\frac{1}{2} = \frac{1}{2} \times \dfrac{10}{10} = \frac{5}{10} = \cdot 5$

Next take $\frac{1}{3}$.

Then $\frac{1}{3} = \frac{1}{3} \times \dfrac{10}{10} = \dfrac{\frac{1}{3} \times 10}{10} = \dfrac{3 + \frac{1}{3}}{10}$

$$= \cdot 3 + \frac{1}{10} \times \frac{1}{3};$$

or $= \frac{1}{3} \times \dfrac{100}{100} = \dfrac{\frac{1}{3} \times 100}{100} = \dfrac{33 + \frac{1}{3}}{100}$

$$= \cdot 33 + \frac{1}{100} \times \frac{1}{3},$$

$$\text{or similarly } = \cdot 333 + \frac{1}{1000} \times \frac{1}{3}.$$

&c. &c.

This shews that the operation will never terminate, or in other words, that $\frac{1}{3}$ can never be *exactly* expressed as a decimal, though by continuing the operation, and taking more and more places of decimals, the difference between $\frac{1}{3}$ and the decimal becomes successively $\frac{1}{30}$, $\frac{1}{300}$, $\frac{1}{3000}$, &c., and will grow less and less; *i. e.*, we can have decimals *approximating* to the value of $\frac{1}{3}$, without ever being exactly equal to it.

From the first of the above two examples, we see that *some* vulgar fractions can be exactly expressed as decimals; and from the second we see that some again cannot be so expressed, though we can have decimals approximating to their value.

115. A Numeration Table for decimals may be given similar to that for integers.

&c.	Hundreds.	Tens.	Units.	Tenths.	Hundredth	&c.

Accordingly, taking as an example, any number, 139·0232, it may be read, one hundred and thirty nine, and two hundredths, three thousandths and two ten thousandths. But 2 hundredths + 3 thousandths + 2 ten thousandths

$$= \frac{2}{100} + \frac{3}{1000} + \frac{2}{10000}$$

$$= \frac{232}{10000},$$

or two hundred and thirty-two ten thousandths. Thus the decimal part may be read in two ways, whereof the latter is the shorter, and being similar to the mode of naming ordinary fractions, by naming separately the numerator and the denominator, is the one usually adopted. Besides these, there is

another common mode of naming decimals, in which the figures in the successive decimal places are read out one after another. Thus the above number will be read thus:—one hundred and thirty-nine, decimal nought, two, three, two.

116. PROP. I. A decimal is multiplied or divided by 10 by removing the decimal point one place towards the right or the left.

Thus, taking any decimal $2{\cdot}307$,

we have $$2{\cdot}307 \times 10 = \left(2 + \frac{307}{1000}\right) \times 10 = \frac{2307}{1000} \times 10$$
$$= \frac{2307}{100} = 23\frac{7}{100} = 23{\cdot}07\,;$$

and $$2{\cdot}307 \div 10 = \frac{2307}{1000} \times \frac{1}{10} = \frac{2307}{10000}$$
$$= {\cdot}2307.$$

PROP. II. The value of a decimal is not altered by affixing ciphers to its right, but is decreased ten-fold by prefixing a cipher to its left.

Thus, taking any decimal ${\cdot}237$, we have

$${\cdot}237 = \frac{237}{1000} = \frac{237 \times 10}{1000 \times 10} = \frac{2370}{10000} = {\cdot}2370.$$

Again, $${\cdot}0237 = \frac{237}{10000} = \frac{237}{10 \times 1000} = \frac{1}{10} \times \frac{237}{1000} = \frac{1}{10} \times {\cdot}237.$$

These effects are very different from the effects of affixing and prefixing ciphers to integers.

Ex. XV.

1. Express in figures the following:—

(1) Three-tenths; seven-tenths; five hundredths; sixty-six hundredths; five and five hundredths; six hundred and sixty and sixty-nine hundredths; one hundred, and one hundredth.

(2) One thousandth; ninety-nine and nine thousandths; one hundred and twenty-three and forty-five thousandths.

(3) One million, and one millionth; five millions and fifty, and two thousand and fifty-three millionths.

2. Express in words the following :—

(1) ·02; 1·03; 21·12; 1·0001; 20·0002.
(2) 123·456; 7891·01112; 13·1517.
(3) ·000001; ·00000050; ·00500.

3. Convert the following decimals into vulgar fractions :—

(1) ·1; ·12; 12·34; 567·8; ·01.
(2) ·0001; ·002; ·03; 100·002.
(3) 35·970; ·00200; 12·321.

4. Express as vulgar fractions in their lowest terms the following :—

(1) ·25; 2·5; ·0025; ·002500.
(2) 56·64; 72·0028; 1·002.
(3) ·128; 17·28; ·0032; 61·64.

5. Express as decimals :—

(1) $\frac{1}{10}$; $\frac{5}{10}$; $2\frac{3}{10}$; $5\frac{6}{20}$; $9\frac{8}{40}$.

(2) $\frac{11}{10000}$; $\frac{32109}{1000}$; $\frac{6814}{100}$; $57\frac{6}{30}$.

(3) $\frac{12345}{500}$; $\frac{123456789}{100000}$; $\frac{100200}{100000}$.

6. Multiply

(1) ·3 by 10, 100, and 1000.
(2) ·003 by 10, 100, and 1000.
(3) 20·00020 by 100 and 10000.
(4) ·156 by 10000 and 100000.
(5) 20·200 by 10 and 100.

7. Divide

(1) ·300 by 100 and 100000.
(2) 3·156 by 100 and 1000.
(3) 3567·1 by 10000 and 10.
(4) 98765·005 by 100 and 1000.
(5) 3·141500 by 100 and 10000.

SECTION VIII. ADDITION OF DECIMALS.

117. RULE. Place the numbers so that units may be under units, tens under tens, &c., and tenths under tenths, hundredths under hundredths, &c.

Commencing with the column of figures on the extreme right, add the numbers as in the Addition of integers, and in the sum, place the decimal point just below the line of decimal points above.

Ex. Add together 237·008, 5·53298, ·023 and 1·61001.

By the Rule we have

$$\begin{array}{r@{\cdot}l} 237 & 008 \\ 5 & 53298 \\ & 023 \\ 1 & 61001 \\ \hline 244 & 17399 \end{array}$$

Reason for the Rule. Since in our Notation, there is a *progressive increase* of local values *tenfold* at each step from right to left, the number of *tens* resulting from the addition of the intrinsic values of the figures in every column must be carried and added to the column to its left; and the figure in the sum below each column will have the local value of that column. In other words, the addition is to be performed as in the case of integers, and the decimal point is to be placed below the line of decimal points above.

The *reason for the Rule* may be also shewn thus:—

$$237{\cdot}008 = 237\frac{8}{1000} = \frac{237008}{1000}.$$

Similarly,

$$5{\cdot}53298 = \frac{553298}{100000};\ \ {\cdot}023 = \frac{23}{1000};\ \text{and } 1{\cdot}61001 = \frac{161001}{100000}.$$

$$\text{Therefore the sum} = \frac{237008}{1000} + \frac{553298}{100000} + \frac{23}{1000} + \frac{161001}{100000}$$

$$= \frac{23700800 + 553298 + 2300 + 161001}{100000}$$

$$= \frac{24417399}{100000} = 244{\cdot}17399.$$

Ex. XVI.

1. Add together

(1) ·12345, 1·2345, 12·345, 123·45, and 1234·5.
(2) 1000·1, 200·02, 30·003, 4·0004, and ·50005.
(3) ·000123, ·0045, ·067, ·89, ·10, and 11.
(4) 123·4589, 1234·56789, and 12345·6789.
(5) 27·0039, ·00009, 1000, 556, and ·00556.
(6) 1·003, 40·0005, 600·00007, and 80000·000009.

2. Find the sum of

(1) Three-tenths; seven hundredths; seventeen thousandths; and two hundred, and fifty-three millionths.

(2) Fifty-five hundredths; sixty-six thousandths; and seventy-seven millionths.

(3) One hundred and twenty thousandths; one million, and one millionth; and fifty thousand, and fifty thousandths.

3. Find the value of

(1) 200·003 + 123·789 + 88·009 + 35·005.
(2) ·0073 + ·173 + 128·359 + 627·047.
(3) 57·68 + 68·79 + 79·810 + 810·911.
(4) ·0047 + ·00059 + ·0000611 + ·00000713.
(5) 1200·3 + 567·27 + 827·447 + 1·2.
(6) 55·55 + 66·66 + 77·77 + 88·88 + 99·99.

SECTION IX. SUBTRACTION OF DECIMALS.

118. Rule. Place the subtrahend below the minuend as in the Addition of decimals.

Affix ciphers to the right of either decimal, if necessary, to make the number of decimal places the same in both.

Then perform the subtraction as in the case of integers, and place the decimal point in the difference below the decimal point above.

Ex. Subtract 2·932 from 26·03

By the Rule we have

$$\begin{array}{r} 26{\cdot}030 \\ 2{\cdot}932 \\ \hline 23{\cdot}098 \end{array}$$

Reason for the Rule. The affixing of ciphers to the right does not alter the value of a decimal (Art. 116). The rest of the reason is the same as that given in the case of the Addition of decimals.

The *reason for the Rule* may be also shewn thus :—

$$26{\cdot}03 - 2{\cdot}932 = \frac{2603}{100} - \frac{2932}{1000}$$
$$= \frac{26030 - 2932}{1000} = \frac{23098}{1000}$$
$$= 23{\cdot}098.$$

Ex. XVII.

1. Subtract

(1) 1·23 from 45·6. (2) 23·45 from 67·89.
(3) 13·3 from 15·27. (4) 29·02 from 30·30.
(5) 41·0056 from 59·9. (6) 356·01 from 1000·0004.

2. Find the difference between

(1) One and one-tenth.
(2) Three and three hundredths.
(3) One and one millionth.
(4) One million and one millionth.
(5) Seven and seven-tenths.
(6) Nine-tenths and nine hundredths.

3. Find the value of

(1) ·0235 – ·008795620.
(2) ·3 – ·00999.
(3) 2·37 – 1·0047.
(4) 53·008 + ·6279 – 7·08.
(5) 7·777 + 99·99 – 11·111.
(6) ·1056 + 5600 – 5·600.

SECTION X. MULTIPLICATION OF DECIMALS.

119. RULE. Multiply the numbers as if they were integers, and in the product, mark off a number of decimal places equal to the sum of the number of decimal places in the multiplicand and the number of decimal places in the multiplier, prefixing ciphers to the left, if necessary.

Ex. 1. Multiply 67·51 by 2·06

By the Rule we have

$$\begin{array}{r} 6751 \\ 206 \\ \hline 40506 \\ 135020 \\ \hline 1390706 \end{array}$$

As the multiplicand has 2, and the multiplier, 2, decimal places, in the product there will be 2 + 2 or 4 decimal places; ∴ the product is 139·0706.

Ex. 2. Multiply ·0027 by 15.

Since 0027 regarded as an integer is the same as 27, we have

$$\begin{array}{r} 27 \\ 15 \\ \hline 135 \\ 27 \\ \hline 405 \end{array}$$

The number of decimal places in the product = 4 + 0 = 4; and we ∴ prefix 1 cipher to 405 to make the number of decimal places 4; and the product required is ·0405.

The *reason for the Rule* will be seen below.

Taking Ex. 1, we have 67·51 × 2·06

$$=\frac{6751}{100}\times\frac{206}{100}=\frac{6751\times 206}{10000}=\frac{1390706}{10000}=139{\cdot}0706.$$

Next taking Ex. 2, we have ·0027 × 15

$$=\frac{27}{10000}\times 15=\frac{27\times 15}{10000}=\frac{405}{10000}=\cdot 0405.$$

These Examples shew that to multiply decimals is the same thing as to multiply them as integers, and then mark off in the product a number of decimal places according to our Rule.

Ex. XVIII.

1. Multiply—

(1) 9·8 by 11·10. (2) 7·65 by 43·21.

(3) ·0023 by 2300. (4) 56 by ·0056.

(5) 357·701 by 3·003. (6) 4729·01 by ·0076.

2. Find the product of

(1) One million and one millionth.
(2) Seven tenths and eight hundredths.
(3) One hundred and one thousandth.
(4) Sixty-seven and sixty-seven hundredths.
(5) One hundred and fifty thousandths.
(6) Three hundredths and six.

3. Find the value of

(1) $3{\cdot}2 \times 32 \times {\cdot}02 \times {\cdot}002$.
(2) $57{\cdot}29 \times 5{\cdot}729 \times {\cdot}0006$.
(3) $127{\cdot}358 \times 359{\cdot}009$.
(4) ${\cdot}00056 \times {\cdot}0067 \times {\cdot}00001$.
(5) ${\cdot}001 \times 200 \times 3{\cdot}30$.
(6) ${\cdot}127 \times 450{\cdot}054 \times {\cdot}09$.

SECTION XI. DIVISION OF DECIMALS.

120. Rule. Perform the division as if the dividend and the divisor were integers.

If the number of decimal places in the dividend equals that in the divisor, the quotient obtained will be the one required; if it exceeds that in the divisor, mark off in the quotient a number of decimal places equal to the difference between the two, prefixing ciphers to the left if necessary; and if it is less than the number of decimal places in the divisor, affix a number of ciphers to the right of the quotient equal to the difference between the two.

If, when regarded as integers, the divisor exceeds the dividend, or if the division does not terminate, then affix ciphers to the right of the dividend, and carry on the operation as

long as necessary, taking care, in pointing the quotient, to regard these ciphers as so many additional places of decimals in the dividend.

Ex. 1. Divide 1·2 by ·025.

By the Rule we have ($\because$ 025 regarded as an integer is the same as 25)

$$\begin{array}{r l}
25) & 12,00\ (48 \\
 & \underline{100} \\
 & \ \ 200 \\
 & \ \ \underline{200}
\end{array}$$

We put a comma to separate the additional ciphers affixed. Here the total number of decimal places in the dividend being 3, *i. e.*, the same as that in the divisor, the quotient required is 48.

Ex. 2. Divide ·272 by 2·9 to 4 places of decimals in the quotient.

By the Rule we have

$$\begin{array}{r l}
29) & 272,00\ (937 \\
 & \underline{261} \\
 & \ \ 110 \\
 & \ \ \underline{87} \\
 & \ \ \ 230 \\
 & \ \ \ \underline{203} \\
 & \ \ \ \ \ 27
\end{array}$$

We need not carry on the division further, as we have now the number of decimal places in the quotient $= 5 - 1 = 4$, the required number, and the quotient required is ·0937....

The reason for the Rule will be seen below.

Taking Ex. 1, We have $1{\cdot}2 \div {\cdot}025 = \frac{12}{10} \div \frac{25}{1000}$

$$= \frac{12}{10} \times \frac{1000}{25} = \frac{12}{25} \times \frac{1000}{10} = \frac{12 \times 100}{25} \times \frac{1}{100} \times \frac{1000}{10} = 12 \times 4 = 48.$$

Next, taking Ex. 2, we have $\cdot272 \div 2\cdot9 = \frac{272}{1000} \div \frac{29}{10}$

$$= \frac{272}{1000} \times \frac{10}{29} = \frac{272}{29} \times \frac{10}{1000} = \frac{27200}{29} \times \frac{1}{100} \times \frac{10}{1000}$$

$$= \frac{27200}{29} \times \frac{1}{10000}.$$

Now $\frac{27200}{29} = 937\ldots\ldots$

$\therefore \quad \frac{27200}{29} \times \frac{1}{10000} = \cdot0937\ldots\ldots$

These Examples shew that in the Division of decimals, we divide the numbers as if they were integers, affixing ciphers to the dividend if necessary, care being taken to place the decimal point in the quotient according to our Rule.

Ex. XIX.

1. Divide

(1) ·5568 by 2·32.
(2) 2·295 by ·0135.
(3) 78·8977 by 29·33.
(4) 66·4488 by 9·9.
(5) 78·78 by ·0026.
(6) ·00624 by 2·08.
(7) 1122·3333 by ·99.
(8) 121416·3 by ·009.
(9) 2122·2 by ·0018.
(10) 6·33 by ·0025.
(11) 33·363 by ·00275.
(12) 94·5 by ·225.

2. Divide to four places of decimals

(1) 247·943 by 13·3.
(2) 1516·17 by ·0023.
(3) 78·007 by ·0135.
(4) ·0023 by 6·69.
(5) 64·005 by 17·6.
(6) ·0008 by 3·3.
(7) 22·25 by 23·8.
(8) ·0025 by ·009.
(9) 135·002 by ·121.
(10) ·007 by ·00073.

3. Find the value of

(1) $2\cdot5 \div \cdot0025$.
(2) $\cdot25 \div \cdot025$.
(3) $\cdot11 \div \cdot0011$.
(4) $\cdot0143 \div 1\cdot3$.
(5) $1710 \div \cdot0171$.
(6) $\cdot0247 \div \cdot0013$.

4. What number multiplied by ·25 will produce 63?
5. What number multiplied by ·225 will produce 126?
6. What number multiplied by the quotient arising from the division of 12 by ·075 will produce 13?

SECTION XII. CONVERSION OF VULGAR FRACTIONS INTO DECIMALS. RECURRING DECIMALS.

121. We have seen in Art. 112, how to convert a decimal into a vulgar fraction. We have also seen in Art. 114, that some vulgar fractions can be converted into decimals consisting of a *finite* number of figures, and others cannot. We will now give a general Rule for the conversion of vulgar fractions into decimals, and ascertain where such conversion is *exactly* possible and where not.

122. *To convert a vulgar fraction into a decimal.*

RULE. Having reduced the fraction to its lowest terms, divide the numerator by the denominator as in the Division of decimals, affixing ciphers to the numerator, and carrying on the division as far as necessary.

Since a fraction denotes the quotient of the numerator by the denominator, the *reason for the Rule* is evident.

Ex. 1. Convert $\frac{2}{125}$ into a decimal.

By the Rule we have

$$\begin{array}{r} 125)\ 2{,}000\ (16 \\ \underline{125} \\ 750 \\ \underline{750} \end{array}$$

$$\therefore \frac{2}{125} = \cdot 016.$$

The following mode will at once indicate the process and the reason for it.

$$\frac{2}{125} = \frac{2}{5 \times 5 \times 5} \times \frac{10 \times 10 \times 10}{10^3} = \frac{2}{5 \times 5 \times 5} \times \frac{5 \times 2 \times 5 \times 2 \times 5 \times 2}{10^3}$$

$$= \frac{2 \times 2 \times 2 \times 2}{10^3} = \cdot 016.$$

Here we resolve the denominator into factors, and multiply the numerator and denominator by such a power of 10, as will enable us to cancel all these factors, *i. e.*, as is divisible exactly by the denominator.

Ex. 2. Convert $\frac{3}{80}$ into a decimal.

Following the second mode, we have

$$\frac{3}{80} = \frac{3}{10 \times 2 \times 2 \times 2} \times \frac{10 \times 10 \times 10}{10^3} = \frac{3 \times 5 \times 5 \times 5}{10^4} = \cdot 0375.$$

Ex. 3. Convert $\frac{5}{18}$ into a decimal.

By the Rule we have

```
18)5,0000(2777
   36
   140
   126
    140
    126
     140
     126
      14
```

$\therefore \frac{5}{18} = \cdot 2777 \ldots\ldots$

123. Prop. I. A proper fraction in its lowest terms can be converted into a *terminating* decimal, only when the denominator is composed *solely* of the factors 2 and 5.

For, the process for converting a fraction into a decimal, as we have seen in Art. 122, consists only in affixing ciphers to the numerator, *i. e.*, multiplying it by some power of 10, and after dividing the result by the denominator, marking off the proper number of decimal places in the quotient. And as, the fraction being in its lowest terms, the numerator and the denominator have no common factor, $\therefore$ the division will terminate and we shall have a terminating decimal, only when the denominator will divide this power of 10 exactly, *i. e.*, when the denominator consists solely of the factors 2 and 5, which are the only factors of which 10 and its powers are composed.

PROP. II. When the division of the numerator of a proper fraction, with ciphers affixed, by the denominator, does not terminate, the figures in the quotient must begin to recur before we have got a number of them equal to the number of units in the denominator.

For, in carrying on the division, as we always bring down the same figure 0 from the dividend to constitute the successive partial dividends, these partial dividends, and therefore the figures in the quotient, will begin to recur as soon as we get a remainder equal to one of the former remainders. Now ∵ every remainder must be less than the divisor, *i. e.*, the denominator, ∴ the different possible remainders are 1,2,&c., up to the integer next below the denominator: and ∴ the *number* of *different* possible remainders = *the denominator minus unity.* And ∵ the numerator, which forms with affixed ciphers the first partial dividend, is also less than the denominator, ∴ before we have got a number of different partial dividends and a corresponding number of figures in the quotient, equal to the number of units in the denominator, one of the former partial dividends must recur, and thence the figures in the quotient will begin to recur.

Thus take as examples $\frac{1}{3}, \frac{3}{7}$, and $\frac{53}{370}$.

We have

```
I.  3) 1·00 (·33
        9
        10
         9
         1

III. 370) 53·00000 (·14324
          370
          1600
          1480
           1200
           1110
             900
             740
             1600
             1480
              120

II.  7) 3·0000000 (·4285714
        28
         20
         14
          60
          56
           40
           35
            50
            49
             10
              7
             30
             28
              2
```

In I, we see that the partial dividends recur from the first, and the figures of the quotient are 33......

In II, the partial dividends recur, after we get all the possible different partial dividends, *viz.*, 10, 20, 30, 40, 50, 60, in the order 30, 20, 60, 40, 50, 10, and the figures in the quotient recur after the sixth, *viz.*, 1, in the order 4 2 &c.

In III, the partial dividends recur after the fourth, and the figures in the quotient recur after the fourth in the order 432 &c., the first figure 1 not recurring at all.

124. DEFS. Non-terminating decimals in which the figures thus recur are called RECURRING or CIRCULATING DECIMALS.

A circulating decimal is called PURE or MIXED according as its figures recur from the first or not.

The set of recurring figures is called the PERIOD or REPETEND.

Thus, ·333..., ·428571428571428571...., are pure circulating decimals, and ·1432432432...., ·2777..., are mixed circulating decimals.

125. Circulating decimals are written by writing the figures only to the end of the first period, and putting dots over the first and the last figures of the period.

Thus, ·333... is written $\cdot\dot{3}$,

·428571428571.... ... $\cdot\dot{4}2857\dot{1}$,

·1432432... ... $\cdot 1\dot{4}3\dot{2}$,

·277... ... $\cdot 2\dot{7}$.

We now proceed to the converse process of converting recurring decimals into vulgar fractions.

126. *To convert a pure circulating decimal into a vulgar fraction.*

RULE. Write the period as the numerator; and for the denominator write as many *nines* as there are figures in the period; and then reduce the fraction to its lowest terms, if necessary.

Ex. 1. Convert $\cdot\dot{3}$ into a vulgar fraction.

By the Rule we have $\cdot\dot{3} = \frac{3}{9} = \frac{1}{3}$.

Ex. 2. Convert $\cdot\dot{4}2857\dot{1}$ into a vulgar fraction.

We have $.428571 = \frac{428571}{999999} = \frac{47619}{111111} = \frac{4329}{10101} = \frac{1443}{3367} = \frac{3}{7}$.

Reason for the Rule. Take Ex. 2.

Let the circulating decimal $\cdot428571428571....$ be represented by the symbol x.

$$
\begin{aligned}
\text{Then} \quad & x = \cdot428571428571... \\
\text{and} \quad 1000000 \times & x = 1000000 \times \cdot428571428571... \\
& = 428571\cdot42857142.... \text{ (Art 116, Prop. I)}; \\
\therefore 1000000 \times x - & x = 428571\cdot428571428571... \\
& \quad - \cdot428571428571..., \\
\textit{i. e.,} \quad 999999 \times & x = 428571. \\
\text{Hence} \quad & x = \tfrac{428571}{999999}.
\end{aligned}
$$

Here, the *artifice* employed is to multiply x and the circulating decimal by such a power of 10 as will make the first period an integer, and then to get rid of the non-terminating part by subtraction.

Next, take Ex. 1, and let $x = \cdot\dot{3}33......$

Then $\because$ the period has only one figure 3, it will be made an integer by multiplication by 10; and we have

$$
\begin{aligned}
10 \times x &= 3\cdot333...... ; \\
\therefore 10 \times x - x &= 3\cdot333...... \\
& \quad - \cdot333......, \\
\text{or } 9 \times x &= 3, \\
\text{and } \therefore \ x &= \tfrac{3}{9}.
\end{aligned}
$$

127. *To convert a mixed circulating decimal into a vulgar fraction.*

RULE. From the decimal down to the end of the first period, subtract the non-recurring part, regarding both as integers, and write the remainder as the numerator; and for the denominator write as many *nines* as there are figures in the period, followed by as many *ciphers* as there are figures in the non-recurring part.

Ex. Convert $\cdot 1\dot{4}3\dot{2}$ into a vulgar fraction.

By the Rule we have $\cdot 1\dot{4}3\dot{2} = \dfrac{1432-1}{9990} = \dfrac{1431}{9990} = \dfrac{159}{1110} = \dfrac{53}{370}$

Reason for the Rule.

Let $x = 1432432\ldots.$

Then, employing the same artifice as in Art. 126, successively to make the decimal to the end of the first period an integer, and the decimal to the end of the non-recurring part an integer, we have

$$10000 \times x = 1432\cdot 432\ldots,$$
$$\text{and } 10 \times x = 1\cdot 432432\ldots,$$

$\therefore$ by subtraction

$$9990 \times x = 1431$$
$$\text{and } \therefore \quad x = \frac{1431}{9990}.$$

128. To perform *accurately* the fundamental operations with recurring decimals, we must reduce them to vulgar fractions, and then perform the operations with these last; and then we can reduce the resulting vulgar fraction to a decimal.

We can however perform these operations *approximately* without reducing circulating decimals to vulgar fractions, as will be seen in the next Section.

Ex. XX.

1. Convert into decimals the following vulgar fractions :—

(1) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and $\frac{1}{32}$.

(2) $\frac{3}{5}$, $\frac{3}{15}$, $\frac{9}{12}$, $\frac{9}{45}$, and $\frac{27}{90}$.

(3) $\frac{1}{5}$, $\frac{1}{25}$, $\frac{1}{125}$ and $\frac{1}{625}$.

(4) $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{8}$, $\frac{3}{10}$, and $\frac{9}{225}$.

(5) $\frac{121}{275}$, $\frac{635}{640}$, $\frac{320}{400}$, and $\frac{76}{190}$.

(6) $\frac{76}{128}$, $\frac{56}{175}$, $\frac{26}{104}$, and $\frac{27}{72}$.

2. Reduce the following vulgar fractions to decimals correctly to 5 places of decimals :—

(1) $\frac{9}{37}$, $\frac{8}{26}$, $\frac{7}{15}$, $\frac{8}{15}$, and $\frac{10}{11}$.

(2) $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$, and $\frac{1}{13}$.

(3) $\frac{2}{15}$, $\frac{3}{17}$, $\frac{4}{19}$, $\frac{5}{21}$, and $\frac{6}{23}$.

(4) $\frac{11}{12}$, $\frac{12}{13}$, $\frac{13}{14}$, and $\frac{14}{15}$.

(5) $\frac{21}{22}$, $\frac{22}{23}$, $\frac{23}{24}$, and $\frac{25}{26}$.

(6) $7\frac{3}{7}$, $8\frac{4}{9}$, $\frac{7}{121}$, and $\frac{8}{169}$.

3. Reduce the following vulgar fractions to reccurring decimals :—

(1) $\frac{3}{7}$, $\frac{5}{9}$, $\frac{9}{11}$, $\frac{11}{13}$, $\frac{13}{14}$, and $\frac{14}{15}$.

(2) $\frac{5}{18}$, $\frac{20}{21}$, $\frac{17}{12}$, and $\frac{8}{14}$.

(3) $\frac{27}{63}$, $\frac{39}{91}$, $\frac{42}{63}$, and $\frac{55}{121}$.

(4) $\frac{63}{81}$, $\frac{15}{70}$, $\frac{16}{700}$, and $\frac{18}{81}$.

(5) $\frac{11}{33}$, $\frac{15}{35}$, $\frac{21}{39}$, and $\frac{153}{187}$.

(6) $\frac{7}{9}$, $\frac{40}{66}$, $\frac{144}{1728}$, and $\frac{5}{300}$.

4. Convert the following decimals into vulgar fractions :—

(1) $\cdot 7$, $\cdot 8\dot{9}$, $\cdot\dot{8}\dot{1}$, $\cdot\dot{9}\dot{3}$, and $\cdot\dot{1}2\dot{3}$.

(2) $5\cdot 7\dot{8}$, $4\cdot 6\dot{2}$, $2\cdot 3\dot{7}$, and $9\cdot 1\dot{7}$.

(3) $1\cdot 36$, $5\cdot 7\dot{3}$, $6\cdot\dot{2}8$, and $\cdot 7\dot{5}$.

(4) $\cdot 72$, $\cdot 8\dot{3}$, $\cdot 8\dot{1}$, $\cdot 8\dot{5}$, and $\cdot 6\dot{7}$.

(5) $\cdot 02$, $\cdot 003\dot{2}$, $\cdot\dot{6}7\dot{9}$, and $\cdot\dot{3}6\dot{9}$.

(6) $\cdot 00\dot{3}$, $\cdot 00\dot{3}$, $\cdot 0\dot{7}$, and $\cdot\dot{0}6\dot{5}$.

SECTION XIII. APPROXIMATE DECIMAL OPERATIONS.

126. The symbol < read *less than*, indicates that the number before it is less than that after it, and the symbol > read *greater than*, indicates that the number before it is greater than that after it.

130. We have seen in *theory*, what millions, billions, trillions, &c., are; but in *practice*, we seldom have to deal with

numbers above hundreds of millions. Thus, the population of India is about 200 millions; and that of the world is estimated at 1110 millions. The annual revenue of all India is less than 60 crores or 600 millions of rupees. The mean distance of the Sun from the Earth is less than 96 millions of miles.

The higher numbers, billions, &c., are very large, indeed so very large, that we can not readily form any adequate conception of their magnitude. Thus, if it is asked, "How much will a trillion of grains of rice weigh?", one who has never considered the question may, considering the smallness of a grain of rice, answer that the weight will be a few maunds, or at the most, a few hundred maunds. But on a little consideration it will be found that, the weight is something much greater. For, 1 maund $=40$ seers, and 1 seer $=80$ tolas; $\therefore$ 1 maund $=80\times40$ tolas $=3200$ tolas; and on counting the number of grains contained in a quantity of rice weighing 1 tola, it will be found, that it never exceeds 1000, so that 1 maund can never contain more than 3200×1000 or 3200000 grains of rice; and $\therefore$ the number of maunds in one trillion of grains of rice will be at least equal to $\frac{1000000000000000000}{3200000}=\frac{10000000000000}{32}=312500000000$, which is an immense quantity.

131. As numbers above millions are very large numbers, and seldom occur in practice, so numbers below millionths are very small, and may be *neglected* in practice without any *appreciable* error in the results of most of our calculations. Thus, taking the ordinary unit of money, 1 rupee, $\frac{1}{100}$th part of 1 rupee $=\frac{64}{100}$ pice which is $<$ 1 pice; $\therefore$ $\frac{1}{1000000}$ of 1 rupee $<\frac{1}{10000}$ of 1 pice, *i. e.* $\cdot000001$ of 1 rupee $<\cdot0001$ of 1 pice, which is almost inappreciable. Hence, in dealing with decimals, where a rupee is the unit, if we reject the decimal places after the 6th, we shall be rejecting what is almost inappreciable. And the same thing may be shewn of other units. Generally, *if we carry on our operations correctly to* 6 *or* 7 *places of decimals, we shall have an approximation sufficient for all practical purposes.*

We now proceed to give Rules for these approximate or contracted operations.

132. Notation.—Rule. Retain the decimal to the required number of places, increasing by 1 the last figure retained if the first figure rejected is greater than 4.

Ex. Write ·30579734 retaining only 5 places of decimals so as to be approximately correct.

By the Rule, increasing 9 by 1, we have 10 in the place of 9 or 80 in the place of 79; and the required decimal is ·30580.

Reason. ·30580 − given decimal ·30579734 = ·00000266; and ·30579734 − ·30579 = ·00000734. But ·00000266 < ·00000734; ∴ ·30580 is *nearer* to the given decimal than ·30579.

133. Addition.—Rule. In each summand retain 2 or 3 more figures than the required number, observing the Rule in Art. 132; perform the addition, and then in the sum retain the required number of places.

The *reason for the Rule* will appear from a comparison of the contracted and full operations given below.

Ex. Add together 2·53789637, 15·00785678, 20·000087654 and ·1000345678, correctly to 5 places of decimals.

Contracted form.	*Full form.*	
2·5378964	2·537896	37
15·0078568	15·007856	78
20·0000877	20·000087	654
·1000346	·100034	5678
37·6458755	37·645875	3718

∴ the sum required is 37·64587.

134. Subtraction.—Rule. Retain 2 or 3 figures more than the required number of places in the minuend and subtrahend, observing the Rule in Art. 132, and then perform the subtraction, and in the difference retain the required number of places.

The reason for the Rule will appear below.

Ex. Subtract ·06 from ·14 so as to be correct to 5 places of decimals.

Contracted form.

$\cdot 1\dot{4} = 1444444$

$\cdot 0\dot{6} = \cdot 0666667$

$\phantom{\cdot 0\dot{6} = }\ \cdot 0777777$

the diff. $= \cdot 07777$

Full form. (Art. 128.)

$$\cdot 1\dot{4} = \frac{1\dot{4} - 1}{90} = \frac{13}{90}$$

$$\cdot 0\dot{6} = \frac{6}{90}$$

$$\therefore\ \cdot 1\dot{4} - \cdot 0\dot{6} = \frac{13}{90} - \frac{6}{90} = \frac{7}{90} = \cdot 0\dot{7}$$

$$= \cdot 07777777....$$

135. MULTIPLICATION.—RULE. Under the multiplicand write the figures of the multiplier in the reverse order, placing the units' figure below that decimal place of the multiplicand to which the operation is to be carried.

Multiply by each figure of the multiplier all the figures of the multiplicand that are above and to the left of itself, neglecting figures to its right except for the purpose of seeing what should be carried and whether the first figure in any partial product is to be increased by 1 according to Art. 132.

Place the first figures of the several partial products in the same vertical line, add those products together, and in the sum mark off the required number of decimal places.

The sum will be the product required, true or nearly true to the required number of places.

The *reason for the Rule* will appear from a comparison of the contracted and full forms of operation given below. It is based on the following considerations:—

Units × units	give	units.
Units × tenths	....	tenths.
Units × hundredths	....	hundredths.
&c.	....	&c.
Tens × units	....	tens.
Tens × tenths	....	units.
Tens × hundredths	....	tenths.
&c.	....	&c.
Tenths × units	....	tenths.
Tenths × tenths	....	hundredths.
Tenths × hundredths	....	thousandths.
&c.	....	&c.

Ex. 1. Multiply 25·7056 by 18·6203 correctly to 4 places of decimals.

Contracted form.	*Full form.*
25·7056	25·7056
302681	18·6203
2570560	77 | 1168
2056448	5141 | 120
154234	154233 | 6
5141	2056448 |
77	257056 |
478·6460	478·6459 | 8368

Explanation of the process. Above 1 there being no figure, we suppose a 0 supplied, which does not alter the value of the multiplicand, and then we put down 1×6, 1×5, &c., successively. In the next line we put the product by 8. In the third line, $6 \times 6 = 36$, from which we carry 3 and add it to 6×5, thus getting 33, and we then increase the 3 in the units' place of 33 by 1, since 6 the figure rejected is > 4; and thus we have 34 of which we put 4 below 8, and then proceed on. In the next line, $2 \times 5 = 10$, so we carry 1, and add it to 2×0 or 0 and put 1 below 4, and then proceed on. Similarly we get the last line. If we look to the partial products in the contracted operation from below upwards, we see that they are the same as the cut off portions of the partial products in the full form from above downwards.

Ex. 2. Multiply $\cdot\dot{3}$ by $\cdot 1\dot{6}$ correctly to 4 places of decimals.

Contracted form.

$\cdot\dot{3} = 0{\cdot}333333\ldots$

$\cdot 1\dot{6} = 0{\cdot}166666\ldots$

Hence by the Rule we have

$$\begin{array}{r} 0{\cdot}333333\ldots \\ 66610 \\ \hline 0333 \\ 200 \\ 20 \\ 2 \\ \hline \cdot 0555 \end{array}$$

Full form. (Art. 128.)

$$\cdot\dot{3} = \frac{1}{3};\ \cdot 1\dot{6} = \frac{16-1}{90} = \frac{1}{6};$$

$$\therefore \cdot\dot{3} \times \cdot 1\dot{6} = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$$

$$= \frac{5}{90} = \cdot 0\dot{5}$$

$$= \cdot 0555\ldots$$

136. Division.—Rule. In the divisor, retain a number of figures equal to the required number of decimal places together with the number of integral places which the quotient must contain, and make this the new divisor; and in the dividend retain a number of figures that will contain this new divisor less than 10 times but not less than once.

Having obtained the first figure of the quotient, to find the next figure, cut off the last figure of the new divisor, and regard the first remainder as the partial dividend. Having obtained the second figure of the quotient, proceed to find its third figure in the same way; and so on.

In multiplying the divisor by each figure of the quotient, take into account figures of the divisor that are cut off to see what should be carried.

The *reason for the Rule* will appear from a comparison of the contracted and full operations given below.

Ex. 1. Divide 8·613452 by 7·35243 correctly to 4 places of decimals.

Contracted form.

```
7·3524,3) 8·6134,52 (1·1715
           73524
           12610
            7352
            5258
            5147
             111
              74
              37
              37
```

Full form.

```
7·35243) 8·6134|52 (1·1715
           73524|3
           12610|22
            7352|43
            5257|790
            5146|701
             111|0890
              73|5243
              37|56470
              36|76215
```

Ex. 2. Divide $\cdot 1\dot{6}$ by $\cdot\dot{3}$.

Contracted form.

```
·333) ·1666 (.5
       1666
```

Full form. (Art. 128.)

$$\cdot\dot{3} = \frac{1}{3}$$

$$\cdot 1\dot{6} = \frac{15}{90} = \frac{1}{6},$$

$$\therefore \text{ quotient} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2} = \cdot 5.$$

Ex. 3. Divide 9·0072 by $\cdot\dot{0}\dot{7}$ correctly to 2 places of decimals.

Contracted form.

```
·070707)9·0072(127·38
         70707
         19365
         14141
          5224
          4949
           275
           212
            63
            57
             6
```

Full form.

$$\text{The quotient} = \frac{90072}{10000} \times \frac{99}{7}$$

$$= \frac{891\cdot7128}{7}$$

$$= 127\cdot387\ldots$$

137. In approximate operations with terminating decimals, the Rules in Arts. 132 to 136 should always be followed. In the case of recurring decimals, it is advisable to carry on the operations after reducing the decimals to vulgar fractions, as in this way we sometimes get very simple results, as we see in Ex. 2 of Art. 136.

138. We shall now compare vulgar fractions with decimals as regards their respective advantages and disadvantages.

Advantages of the use of Vulgar Fractions.

I. Vulgar fractions occur to us more readily and naturally than the corresponding decimals. Thus $\frac{1}{2}$ occurs to us more readily than $\frac{5}{10}$ or ·5 ; $\frac{1}{4}$, more readily than ·25 ; $\frac{1}{8}$, much more readily than ·125 ; and there is no comparison between $\frac{1}{3}$ and ·333..., $\frac{1}{6}$ and ·1666..., &c., in point of simplicity.

II. By vulgar fractions, we can exactly express all fractional parts with a finite number of figures ; but this is not possible with decimals. Thus, the fractional part *one-third* can be expressed as a vulgar fraction by $\frac{1}{3}$, but as a decimal it will be expressed by ·333......*ad infinitum.*

Advantages of the use of Decimals.

I. The notation of decimals being only an extension of the Common System of Notation, is far more simple and convenient than the notation of vulgar fractions.

Though we cannot express some vulgar fractions *accurately* except by an infinite number of decimal places, yet by taking a sufficient number of these places, we can express them *by an approximation sufficient for all practical purposes.*

II. The fundamental operations are performed in the case of decimals far more easily than in the case of vulgar fractions. Further, as the student will hereafter see, decimals are incomparably better adapted for *Logarithmic* computation than vulgar fractions.

Ex. XXI.

1. Write the following decimals retaining only 5 places of decimals, so as to be approximately correct :—

(1) ·06254678 ; ·123456789 ; ·987654321.

(2) ·135791113 ; ·24681012 ; ·5115253.

(3) ·8642867 ; ·13934771 ; ·72336872.

(4) $\cdot 27\dot{3}$; $\cdot 02\dot{7}\dot{3}$; $\cdot 0\dot{9}2\dot{3}$.

2. Find the value (correct to 5 places of decimals) of

(1) 12·3456789 + 23·4567891 + 34·5678912.

(2) ·0036912 + ·03691215 + ·396121518.

(3) $3\cdot\dot{3} + 2\cdot\dot{5}\dot{7} + 1\cdot 52\dot{4}\dot{7} + \cdot 0\dot{2}$.

(4) $\cdot 34\dot{5} + \cdot 4\dot{5} + 6\cdot\dot{8}\dot{1} + 56\cdot 05\dot{6}$.

(5) $12\cdot 34\dot{5} - \cdot 02\dot{7} + 72 - \cdot\dot{1}2\dot{3}$.

(6) $9\cdot 9\dot{5} + 6\cdot 3\dot{8} - 1\cdot 1\dot{6} - 11\cdot\dot{6}$.

(7) 12·3456789 − 9·87654321

(8) 13·5791113 × 24·681012.

(9) 5·19152925 × 39·354945.

(10) $\cdot\dot{3} \times 4$; $2\cdot 2\dot{7} \times \cdot 1\dot{3}$; $\cdot\dot{1}\dot{5} \times \cdot 2\dot{1}$.

(11) $\cdot 2\dot{4}\dot{7} \times \cdot 5$; $2\cdot 22 \times \cdot 1\dot{8}$; $2\cdot\dot{4} \times 5$.

(12) $\cdot\dot{2} \times 4$; $1\cdot\dot{4} \times 2\cdot\dot{5}$; $6\cdot\dot{7} \times 9\cdot\dot{8}$.

(13) $1234\cdot56789 \div 987\cdot654321$.

(14) $357\cdot9113 \div 246\cdot81912$.

(15) $\cdot\dot{3} \div 6$; $\cdot\dot{6} \div 3$; $\cdot1\dot{6} \div 9$; $15 \div \cdot\dot{3}$.

MISCELLANEOUS QUESTIONS AND EXAMPLES.

139. In working out Examples in decimals, the remarks in Arts. 76 and 77 should be borne in mind.

Ex. 1. Simplify $\dfrac{\cdot1 + \cdot02 \times \cdot8}{5 \div \frac{1}{2} - \cdot\dot{3}} + \dfrac{3\frac{3}{4}}{\cdot3}$.

The given expression

$$= \frac{\cdot1 + \cdot016}{10 - \frac{8}{9}} + \frac{\frac{15}{4}}{\frac{3}{10}} = \frac{\frac{116}{1000}}{\frac{29}{3}} + \frac{25}{2}$$

$$= \frac{116}{1000} \times \frac{3}{29} + \frac{25}{2} = \frac{12}{1000} + \frac{125}{10}$$

$$= \cdot012 + 12\cdot5 = 12\cdot512.$$

Ex. 2. What decimal added to $\frac{1}{2} + \frac{1}{3} + \cdot4 + \cdot5$ will make the sum equal to 6?

By the question,

the decimal required $= 6 - (\frac{1}{2} + \frac{1}{3} + \cdot4 + \cdot5)$ expressed as decimal.

Now $6 - (\frac{1}{2} + \frac{1}{3} + \cdot4 + \cdot5) = 6 - \left(\frac{5}{6} + \frac{9}{10}\right)$

$$= 6 - \frac{52}{30}$$

$$= 6 - \frac{26}{15}$$

$$= \frac{64}{15}$$

$$= 4\cdot2\dot{6};$$

$\therefore$ $4\cdot2\dot{6}$ is the decimal required.

Ex. 3. Find the value of $1 + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} +$ &c. to 5 places of decimals.

We have
$$1=1$$
$$\frac{1}{1}=1$$
$$\frac{1}{1\times2}=\cdot5$$
$$\frac{1}{1\times2\times3}=\cdot1666666\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4}=\cdot0416666\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4\times5}=\cdot0083333\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4\times5\times6}=\cdot0013888\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4\times5\times6\times7}=\cdot0001984\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4\times5\times6\times7\times8}=\cdot0000248\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4\times5\times6\times7\times8\times9}=\cdot0000027\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}=\cdot0000002\ldots\ldots\ldots\ldots\ldots$$
$$\&c. = \&c.$$

$\therefore$ the sum $=2\cdot7182814$;

and $\therefore$ the value required $=2\cdot71828\ldots\ldots\ldots\ldots$

Here, to ensure accuracy we retain 7 places of decimals (*i. e.*, 2 more than the number required) in the values of the several fractions, and we stop after $\dfrac{1}{1\times2\times3\ldots\ldots\ldots\times10}$ as the first seven figures in the values of all the succeeding fractions will be zeros.

Ex. XXII.

I.

1. What is a Decimal Fraction, and why is it so called?

Shew that the ordinary Notation of decimals is an extension of the Common System of Notation.

2. Shew how to convert a decimal into a vulgar fraction.

Convert $\cdot 0325$ into a vulgar fraction in its lowest terms.

3. Shew that a decimal is multiplied or divided by any power of 10 by removing the decimal point to the right or to the left a number of places equal to the index of the power.

Find the value of $\dfrac{195\cdot 0035 \times 10^5}{10^3} - \cdot 05 \div 10^3 + \cdot 25 \times 10^6$.

4. Convert the value of $\frac{9}{32} + \frac{6}{125} - \frac{3}{24}$ into a decimal.

5. A number being multiplied by 1000 has 2 places of decimals in the product. How many places of decimals had it originally?

6. A number being divided by 10^4 has 6 places of decimals in the quotient. Find the number of decimal places in the dividend.

II.

1. State the Rule for the Addition of decimals.

Add together $3\cdot 25$, $\cdot 0928$, $\cdot 0013$, and $5\cdot 031$.

2. Give the reason for the Rule for the Subtraction of decimals.

Find the difference between

$$\cdot 1234 + 1\cdot 234 + 12\cdot 34 + 123\cdot 4$$

and $$1234 + 1234 + 1234 + 1234.$$

3. Find the value of

$$1000 - \cdot 0001 + 1\cdot 000 - \cdot 100000.$$

4. Point out the effects of affixing and prefixing ciphers to a decimal.

Find the difference between $\cdot 375$ and $\cdot 0375$, and between $\cdot 003750$ and $\cdot 00375$.

5. A person owns $\frac{3}{8}$ of an estate, and he subsequently purchases $\cdot 24$ of the same. What decimal of the whole estate does he now possess, and what decimal of it must be dispose of, that he may have exactly half of the estate left for himself?

6. Find the sum of the sum and the difference of $\cdot 2357$ and $2\cdot 357$ without actually performing the operations of Addition and Subtraction.

III.

1. Give the reason for the Rule for the Multiplication of decimals.

Multiply $\cdot 035$ by $5\cdot 23$, and find the difference between the result obtained and the product of 523 and 35.

2. The sum of the ages of two boys is 15 years, and the age of one of them is $\cdot 5$ of the other. What is the age of each?

3. The greater of two numbers is $1\cdot 75$ times the less, and their difference is equal to $\cdot 54 \div \cdot 09$. Find the numbers.

4. A gentleman spends a certain sum in the first week of a month; $1\cdot 5$ times that sum in the second week; and in the third week, $2\cdot 16$ times the amount spent in the first two weeks together; and he finds that he has altogether spent 790 rupees. How much did he spend in each week?

5. Find the value of $\dfrac{2 \times \cdot 25 - \frac{3}{4} \text{ of } \cdot 16}{1\cdot 9}$.

6. Divide 1500 rupees amongst three persons A, B, and C, in such a manner that B may get $\cdot 35$ of what A gets, and C, $\frac{3}{7}$ of what B receives.

IV.

1. State the Rule for the Division of decimals. Divide $6\cdot 25$ by 25, $2\cdot 5$, and $\cdot 0025$.

2. What number must be multiplied by $\cdot 023$ to produce 1610?

3. What number must be divided by $\cdot 029$ to produce $1\cdot 6$?

4. Find the greatest and the least of the following three numbers:—

$\cdot 5 \times \cdot 02$; $\cdot 25 \times \cdot 10$; and $1\cdot 05 \div 21$.

5. Find the value of
$\frac{1}{2} + \frac{3}{4}$ of $\cdot 264 \times \cdot 05 + \cdot 625 \div \frac{1}{2} - \cdot 0001$.

6. What decimal of 2·75 is the number 2·2?

V.

1. What is the Recurring Decimal, and why is it so called? Shew how to convert a recurring decimal into a vulgar fraction.

2. Shew that when a vulgar fraction cannot be converted into a terminating decimal, the figures of the decimal must recur.

Convert $\frac{2}{17}$ into a decimal.

3. Is the system of notation for decimal fractions sufficient for the expression of all fractions?

Shew that the only vulgar fractions that can be converted into terminating decimals are those in which the denominators are composed exclusively of the factors 2 and 5.

Can the fraction $\frac{15}{768}$ be converted into a terminating decimal?

4. Multiply $\cdot\dot{2}7\dot{9}$ by ·45 and divide the result by ·31.

5. Shew that the decimal 3·1416 lies between $\frac{22}{7}$ and $\frac{355}{113}$.

6. Convert $\cdot\dot{3}7\dot{8}$ and $\cdot 5\dot{2}\dot{7}$ into vulgar fractions, and $\frac{2}{7}$ and $\frac{9}{16}$ into decimals.

VI.

1. Find the value (correct to 4 places of decimals) of

$$\frac{1}{2} - \frac{1}{3 \times 2^3} + \frac{1}{5 \times 2^5} - \frac{1}{7 \times 2^7} + \&c.$$
$$+ \frac{1}{3} - \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} - \frac{1}{7 \times 3^7} + \&c.$$

2. Explain clearly how the operations of Addition and Subtraction may be performed with recurring decimals, without converting them into vulgar fractions, so as to give results that are approximately correct.

Find the value of

$$\cdot\dot{3} + \cdot 3\dot{8} - 02\dot{7} + 7\cdot\dot{8} - 2\cdot 5.$$

3. Find the value of

$$1 + \frac{1}{2 \times 10} + \frac{2}{3 \times 10^2} + \frac{3}{4 \times 10^3} + \&c.$$

correctly to 5 places of decimals.

4. The first of three numbers is $\cdot\dot{3}$ of the second, and the second, $\cdot\dot{4}$ of the third; and their sum is 43. Find the numbers

5. Find the value of

$$4 \times \left\{ \frac{1}{5} - \frac{1}{3 \times 5^3} + \frac{1}{5 \times 5^5} - \frac{1}{7 \times 5^7} + \&c. \right\}$$
$$- \left\{ \frac{1}{239} - \frac{1}{3 \times 239^3} + \frac{1}{5 \times 239^5} - \frac{1}{7 \times 239^7} + \&c. \right\}$$

correctly to 4 places of decimals.

6. Express $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$ as a decimal correctly to 5 places of decimals.

CHAPTER III.

THE FUNDAMENTAL OPERATIONS WITH CONCRETE INTEGERS.

SECTION I. TABLES OF MONEY, WEIGHTS, AND MEASURES.

140. We have hitherto considered our operations with regard to abstract numbers, or concrete numbers of one denomination only. But we have also to deal with concrete numbers of different denominations. Thus, we may be required to add together several sums of money, each consisting of rupees, annas, and pies; or we may have to find the difference between two sums each consisting of rupees, annas, and pies; and so on.

141. The different kinds of concrete numbers which occur in Arithmetic are those expressing Values of Money, Weights of Substances, and Measures of Length, of Surface, of Solidity, of Capacity, of Angular Magnitudes, of Number, and of Time. For the numerical representation of quantities of each of these different kinds, a certain quantity of the same kind is taken as the unit of the highest denomination, so that any given quantity would be represented by the number indicating the number of times that it contains the unit of the same kind; and the unit of each kind is divided and subdivided into units of lower denominations, the object being to enable us to express quantities in *integers* of different denominations. Thus, suppose we have to express numerically the length of a line which contains the linear unit 1 foot 5 times together with $\frac{1}{2}$ of a time; the whole length expressed in feet will be represented by the mixed number $5\frac{1}{2}$; and if we divide the length of 1 foot into 12 equal parts, and call each part an inch, the *fraction* $\frac{1}{2}$ of a *foot* will be represented by the *integer* 6 *inches*, and the whole length, by 5 feet 6 inches.

The unit adopted in each case and its successive divisions and subdivisions, are purely conventional, and are different in different countries; though some grounds of convenience must form the basis of these conventions.

We subjoin the ordinary English and Indian Tables connecting the units of different denominations for each of the above kinds of concrete quantity, with short remarks on each.

ENGLISH TABLES.

Table of Money.

142. In this Table, the different units are connected thus :—

4	Farthings	make	1 Penny	written	1*d.*
12	Pence	......	1 Shilling	......	1*s.*
20	Shillings	......	1 Pound	......	£1.

The symbols £. *s.* *d.* and *q* (the former symbol for a farthing) are the initials of the words *libra. solidus, denarius,* and *quadrans,* Latin names of certain Roman coins or sums of money. The symbol *q* is not now in common use, the fractions $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$, annexed to pence, representing 1, 2, and 3 farthings respectively.

The following are the coins now current in England :—

Copper Coins.

A Farthing, the coin of least value.
A Half-penny = 2 Farthings.
A Penny = 4 Farthings.

Silver Coins.

A Threepenny-piece	= 3*d.*
A Fourpenny-piece	= 4*d.*
A Sixpence	= 6*d.*
A Shilling	= 12*d.*
A Florin	= 2*s.*
A Half-Crown	= 2*s.* 6*d.*
A Crown	= 5*s.*

Gold Coins.

A Half-sovereign	= 10*s.*
A Sovereign	= 20*s.*

The following are some of the old English coins which are not now in use :—

Silver Coins.

A Groat	= 4*d.*
A Tester	= 6*d.*

Gold coins.

		£.	s.	d.
A Noble	=	0	6	8.
An Angel	=	0	10	0.
A Half guinea	=	0	10	6.
A Mark or Merk	=	0	13	4.
A Guinea	=	1	1	0.
A Carolus	=	1	3	0.
A Jacobus	=	1	5	0.
A Moidore	=	1	7	0.

The standard of gold coin in England is 22 parts of pure gold and 2 parts of copper melted together. Each of these 24 parts is termed a *carat*, and the standard gold is said to be 22 carats fine. The weight of a Sovereign $= \frac{40}{1869}$ of 1 pound Troy. (See Art. 143.)

The standard of silver coin is 37 parts of pure silver and 3 parts of copper. The weight of a shilling $= \frac{1}{66}$ of 1 pound Troy. The weight of a penny $= \frac{1}{24}$ of 1 pound Avoirdupois. (See Art. 145.)

In England, the copper coinage is not a *legal tender* for more than 12*d.*; nor is the silver coinage for more than 40*s.*; the gold coinage being a *legal tender* for all amounts.

Table of Troy Weight.

143. The name of this Table is derived by some from *Troyes* a city in France, and by others from *Troynovant* a name for London. It is used in weighing gold, silver, and other costly articles, and also in philosophical investigations.

In this Table,

24 Grains (grs.) make	1 Pennyweight	1	dwt.
20 Penny weights	1 Ounce	1	oz.
12 Ounces	1 Pound	1	lb. or ℔.

Diamonds and other precious stones are weighed by *carats*, each carat weighing $3\frac{1}{6}$ grains.

Table of Apothecaries' Weight.

144. *In this Table which is used in weighing medicines,*

20 Grains (grs.)	make	1 Scruple	....	1 sc. or ℈i
3 Scruples	....	1 Dram	....	1 dr. or ʒi
8 Drams	....	1 Ounce	....	1 oz. or ℥i
12 Ounces	....	1 Pound	....	1 lb. or ℔.

In this Table, the pound, and consequently the ounce and the grain, are the same as in Troy Weight.

Table of Avoirdupois Weight.

145. The word Avoirdupois is derived from *Avoirs* (goods) and *Poids* (weight). Avoirdupois weight is used in weighing all heavy and coarse articles.

In this Table,

16 Drams	make	1 Ounce	....	1 oz.
16 Ounces	....	1 Pound	....	1 lb.
28 Pounds	....	1 Quarter	....	1 qr.
4 Quarters	....	1 Hundredweight	....	1 cwt.
20 Hundredweights	....	1 Ton	....	1 Ton.

1 Stone of meat or fish $= 8$ lbs.
1 Stone (generally) $= 14$ lbs.
1 Firkin of Butter $= 56$ lbs.
1 Fodder of Lead $= 19\frac{1}{2}$ cwt.
1 Pack of Wool $= 240$ lbs.
1 lb. Avoirdupois $= 7000$ grs. Troy.
1 lb. Troy $= 5760$ grs. Troy.
1 lb. Avoirdupois $= \frac{7000}{5760}$ ℔s. Troy $= \frac{175}{144}$ of 1℔. Troy.

Table of Lineal Measure.

146. In this measure, used in measuring distances, lengths, breadths, and the like,

3 Barleycorns	make	1 Inch	...	...	1 in.
12 Inches	...	1 Foot	...	...	1 ft.
3 Feet	...	1 Yard	...	...	1 yd.
$5\frac{1}{2}$ Yards	...	1 Rod, Pole or Perch			1 po.
40 Poles	...	1 Furlong	...	...	1 fur.
8 Furlongs	...	1 Mile	...	...	1 m.
3 Miles	...	1 League	...	...	1 lea.
$69\frac{1}{2}$ Miles	...	1 Degree	...		1 deg. or 1°.

4 Inches	make	1 Hand (used in measuring horses)
22 Yards	...	1 Chain } (...............land)
100 Links	...	1 Chain }

1 Palm = 3 in., 1 Span = 9 in., 1 Cubit = 18 in.
1 Pace = 5 ft., 1 Geographical Mile = $\frac{1}{60}$th of a degree.
1 Line = $\frac{1}{12}$th of an inch.
1 Mile = 1760 yards.

Table of Cloth Measure.

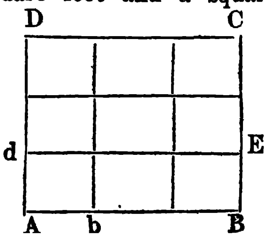
147. In this measure,

$2\frac{1}{4}$ Inches	make	1 Nail.		
4 Nails	...	1 Quarter	...	1 qr.
4 Quarters	...	1 Yard	...	1 yd.
5 Quarters	...	1 English Ell.		
6 Quarters	...	1 French Ell.		
3 Quarters	...	1 Flemish Ell.		

Table of Superficial or Square Measure.

148. In this measure, which is used in measuring areas, the units of the first four denominations, *viz.*, the Square Inch, the Square Foot, the Square Yard, and the Square Pole, are areas enclosed by squares having for their sides an Inch, a Foot, a Yard and a Pole of lineal measure; and as the lineal inch, foot, &c., have already got with one another the *conventional* relations given in Art. 146, the square inch, square foot, &c., must have with one another, certain relations which are not *conventional* but *necessary*. We must find out these relations before we give the Table.

To find the relation between a square foot and a square yard, take ABCD to represent a square yard, so that A B = A D = 1 lineal yard; and divide A B, A D each into 3 equal parts like Ab, Ad; then each of these parts = 1 foot. Draw lines horizontally and vertically as in the figure; then each little square is 1 square foot.



Thus the no. of sq. feet in 1 sq.yard
= ....small squares in the large square
= ....horizontal rows like A B E d
× ....small squares in each row
$= 3 \times 3 = 9$.

Similarly it may be shewn that the number of sq. inches in 1 sq. foot $= 12 \times 12$; & so on.

Our Table will therefore run thus :—

144 Square Inches	make	1 Square Foot	...	1 sq. ft.
9 Square Feet	...	1 Square Yard	...	1 sq. yd.
$30\frac{1}{4}$ Square Yards	...	1 Square Pole	...	1 sq. po.
40 Square Poles	...	1 Rood	...	1 ro.
4 Roods	...	1 Acre	...	1 ac.

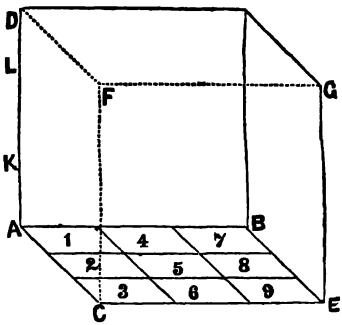
1 Ac. = 4 roods = 4×40 sq. poles
= $4 \times 40 \times 30\frac{1}{4}$ sq. yds.
= 4840 sq. yds.
1 Sq. Chain = (22×22) sq. yds.
= 484 sq. yds.
Hence 10 Sq. Chains = 1 ac.
1 Sq. Mile = 1760×1760 sq. yds. = 640 acres.
A Rod of Brickwork = $272\frac{1}{4}$ sq. ft.
A Square of Flooring = 100 sq. ft.
A Yard of Land = 30 ac.
A Hide of Land = 100 ac.

Two square feet = twice the area of one square foot, but *two feet square* = area enclosed by a square having two feet for its side = 2×2 or 4 square feet. This distinction should be borne in mind.

Table of Solid or Cubic Measure.

149. In this measure, which is used in measuring volumes, the successive units viz. the Cubic Inch, the Cubic Foot, and the Cubic Yard, are cubes having for their edges the lineal Inch, Foot, and Yard, respectively. Therefore, for the same reason as in Art. 148, the cubic inch, cubic foot and cubic yard must have with one another certain *necessary* relations, which may be ascertained thus :—

Let the annexed figure represent a cubic yard, then A B = A C = A D = 1 lineal yd., and the face ABEC is a sq. yd. containing, as in the figure, 9 sq. feet. Through the lines of division of ABEC, draw planes parallel to ACFD, and CEGF; then these planes by their intersection will divide the cube into 9 equal solid figures each standing on one of the 9 squares in ACEB. If now we draw through K and L (the points which divide AD into 3 equal parts) planes parallel to ABEC, we shall have each of these 9 solids divided into 3 small cubes, each of which is a cubic foot; and thus in a cubic yard there are altogether 3 × 9 or 27 cubic feet.



Similarly a cubic foot = 12 × 12 × 12 or 1728 cubic inches. Our Table will therefore stand thus:—

1728 Cubic Inches make	1 Cubic Foot	...	1 cub. ft.
27 Cubic Feet	1 Cubic Yard	...	1 cub. yd.

A Load of Rough Timber = 40 cub. ft.
A Load of Squared Timber = 50 cub. ft.
A Ton of Shipping = 42 cub. ft.

Table of Wine Measure.

150. In this measure by which wines and all liquids except malt liquors and water are measured,

4 Gills	make	1 Pint ...	1 pt.
2 Pints		1 Quart ...	1 qt.
4 Quarts		1 Gallon ...	1 gal.
10 Gallons		1 Anker ...	1 ank.
18 Gallons		1 Runlet ...	1 run.
42 Gallons		1 Tierce ...	1 tier.
2 Tierces		1 Puncheon	1 pun.
63 Gallons		1 Hogshead	1 hhd.
2 Hogsheads		1 Pipe ...	1 pipe.
2 Pipes		1 Tun ...	1 tun.

TABLE OF ALE AND BEER MEASURE.

151. In this measure used in measuring malt liqu water,

2 Pints	make 1	Quart ...	1 qt.
4 Quarts	... 1	Gallon...	1 gal.
9 Gallons	... 1	Firkin...	1 fir.
2 Firkins	... 1	Kilderkin	1 kil.
2 Kilderkins	... 1	Barrel ...	1 bar.
$1\frac{1}{2}$ Barrels or 54 Gallons	... 1	Hogshead	1 hhd.
2 Hogsheads	... 1	Butt ...	1 butt.
2 Butts	... 1	Tun ...	1 tun.

TABLE OF CORN OR DRY MEASURE.

152. In this measure,

2 Quarts	make 1	Pottle ...	1 pot.
2 Pottles	... 1	Gallon ...	1 gal.
2 Gallons	... 1	Peck ...	1 pk.
4 Pecks	... 1	Bushel ...	1 bus.
2 Bushels	... 1	Strike ...	1 str.
2 Strikes	... 1	Coomb ...	1 coomb.
2 Coombs	... 1	Quarter ...	1 qr.
5 Quarters	... 1	Load ...	1 load.
2 Loads	... 1	Last ...	1 last.

TABLE OF COAL MEASURE.

153. In this measure used in measuring coals, lime, &c.,

4 Pecks	make 1	Bushel.
3 Bushels	... 1	Sack.
12 Sacks	... 1	Chaldron.

TABLE OF ANGULAR MEASURE.

154. The circumference of every circle is supposed to be divided into 360 equal parts, each of which is called a degree. The degree and its subdivisions are connected thus :—

60 Seconds	make	1 minute ...	$1'$.
60 Minutes	...	1 Degree ...	1°.

1 Second is written $1''$.

Table of Number.

155.

12 Units	... make	1	Dozen.
12 Dozen	...	1	Gross.
20 Units	...	1	Score.
120 Units	...	1	Long Hundred.
24 Sheets of Paper	...	1	Quire.
20 Quires	...	1	Ream.
10 Reams	...	1	Bale.

Table of Time.

156.

60 Seconds	... make	1	Minute	...	1′
60 Minutes	...	1	Hour	...	1 hr.
24 Hours	...	1	Day	...	1 day
7 Days	...	1	Week	...	1 wk.

1 Second is written 1″.

A year is divided into 12 parts called Calendar Months, each containing a certain number of days as given in the following lines:—

Thirty days have September,
April, June, and November:
February has twenty-eight alone,
And all the rest have thirty-one:
But leap-year coming once in four,
February then has one day more.

The time from one vernal equinox to another, that is the solar year, consists of 365·242218 *mean* solar days; the ordinary year of 365 *mean* solar days differs therefore from the solar year by·242218 or nearly $\frac{1}{4}$ of a day; and in 4 years this difference would nearly amount to a day. To prevent this *error*, Julius Cæsar introduced a method of correction by which every fourth year called a *Leap* or *Bissextile* year is made to consist of 366 days, an extra day, called the *Intercalary* day, being added to the month of February.

Thus, the average Julian year = 365·25 days;
but the solar year = 365·242218 days
∴ the difference = ·007782 of a day;

and this difference in 400 years would amount to 400 × ·007782 or 3·1128 days; that is, if in any year, the equinox fall on a

certain day, 400 years after that year, it will fall 3 days earlier, and 1257 years after, it will fall $1257 \times \cdot 007782$ or $9 \cdot 781974$ days, *i. e.*, nearly 10 days earlier, according to the Julian Calendar. This in fact was what was actually observed in the year 1582 of the Christian Era; for whereas in the year 325, the year of the Council of Nice, the vernal equinox fell on the 21st of March, in the year 1582, or 1257 years after, it fell on the 11th of March. To correct this error, Pope Gregory XIII caused 10 days to be omitted in that year, making the 15th of October follow the 4th; and to prevent the recurrence of this error in future, he ordered that in every 400 years, three of the Julian leap years should be regarded as ordinary years, *viz.*, those which completec enturies, the numbers expressing which centuries are not multiples of 4: thus, in the Gregorian or New Style, 1700,1800, and 1900, are not leap years, but 1600, and 2000 are.

In England, the New Style was adopted on the 2nd of September 1752 when the error amounted to 11 days.

157. An Act of Parliament "FOR ASCERTAINING AND ESTABLISHING UNIFORMITY OF WEIGHTS AND MEASURES" came into operation on the 1st of January 1826; enacting,

First: That the *brass Standard Yard* of 1760, then in the custody of the Clerk of the House of Commons, shall be the *Imperial Standard Yard* (the brass being at the temperature of 62° by Fahrenheit's thermometer); and that it shall be the only standard measure of extension wherefrom all others are to be deduced; and that the $\frac{1}{36}$th part of this yard shall be an inch.

Now the length of the pendulum vibrating seconds in the latitude of London in a vacuum, and at the level of the sea, is found to be $39 \cdot 1393$ such inches. This affords the means of recovering the Standard Yard should it be lost.

Secondly: That the *brass weight of one Pound Troy* of 1758, then in the custody of the same officer, shall be the *Standard Measure of Weight*, and that the $\frac{1}{5760}$th part of it shall be a grain, and that 7000 such grains shall be contained in one Pound Avoirdupois.

Now the weight of a cubic inch of distilled water is 252·458 grains Troy, the barometer being at 30 inches and the thermometer at 62°. This affords the means of recovering the Imperial Standard Pound if it be lost.

Thirdly: That the *Standard Measure of Capacity* shall be the *Imperial Standard Gallon* containing 10 Pounds Avoirdupois weight of distilled water, the barometer being at 30 inches and the thermometer at 62°.

Now this weight fills 277·274 cub. in. and thus the Imperial Standard Gallon contains 277·274 cub. in.

158. An Act of Parliament was passed in 1864, legalizing the use of the Metric System of Weights and Measures.

In this system, the several Tables run thus :—

I. MONEY. 1 FRANC (the unit) = about $9\frac{3}{4}d$.

10 Centimes (c.)	make	1	Decime.
10 Decimes		1	FRANC.

II. WEIGHT. 1 GRAM (the unit) = 15·4323487 grs.

10 Milligrams	make	1	Centigram.
10 Centigrams	...	1	Decigram.
10 Decigrams	...	1	GRAM.
10 Grams ...	...	1	Dekagram.
10 Dekagrams	...	1	Hectogram.
10 Hectograms	...	1	Kilogram ($=2\frac{1}{5}$ lbs. Avoir. nearly).
10 Kilograms	...	1	Myriagram.

III. LENGTH. 1 METRE (the unit) = 39·3708 inches.

10 Millimetres	make	1	Centimetre.
10 Centimetres			Decimetre.
10 Decimetres			METRE.
10 Metres ...			Dekametre.
10 Dekametres			Hectometre.
10 Hectometres			Kilometre (= mile nearly).
10 Kilometres		1	Myriametre.

IV. SURFACE. 1 ARE (the unit) = 100 sq. metres.
= 119·6033 sq. yds.

10 Centiares	make	1 Deciare.
10 Deciares	...	1 ARE.
10 Ares	...	1 Dekare.
10 Dekares	...	1 Hectare.

V. SOLIDITY. 1 STERE (the unit) = 1 cub. metre
= 35·317 cub. ft.

10 Decisteres	make	1 STERE.
10 Steres	...	1 Dekastere.

VI. CAPACITY. 1 LITRE (the unit) = $\frac{1}{1000}$th part of 1 cub. metre
= 1·76077 pints.

10 Centilitres	make	1 Decilitre.
10 Decilitres	...	1 LITRE.
10 Litres	...	1 Dekalitre.
10 Dekalitres	...	1 Hectolitre.
10 Hectolitres	...	1 Kilolitre.

It will be observed that the several multiples and submultiples of the unit in each of the foregoing Tables, results from the multiplication and division of the unit by powers of 10.

INDIAN TABLES.

TABLE OF MONEY.

159. In this Table,

3 Pies	make	1 Pice.
12 Pies or 4 Pice	...	1 Anna.
16 Annas	...	1 Rupee

15 Sicca Rupees = 16 current Rupees.
The following are the coins now in use.

Copper Coins.

A Pie.
A Half pice.
A Pice.
A Double pice.

Silver Coins.

A Two-anna piece.
A Four-anna piece.
A Half rupee.
A Rupee.

The following are the only *Gold Coins* allowed to be coined under the Indian Coinage Act 1870:—

A Five-rupee piece.
A Ten-rupee piece.
A Gold Mohur or Fifteen-rupee piece.
A Double Gold Mohur or Thirty rupee piece.

The gold mohur weighs 180 grs. Troy and consists of 11 parts of pure gold and 1 part of alloy. The other gold coins are of the same fineness, and their weights are proportional to their values

The rupee weighs 180 grs. Troy and consists of 11 parts of pure silver and 1 part of alloy.

The other silver coins are of the same fineness and of proportionate weight.

The double pice weighs 200 grs. Troy The other copper coins are of proportionate weight.

Under the Indian Coinage Act (Act XXIII of 1870) now in force, the gold coinage is not a legal tender; the rupee and the half rupee are a legal tender for any amount; and the copper and the other silver coins are a legal tender only for fractions of a rupee.

Approximately, 1 rupee = 2 shillings. But the exact value of a rupee at any time depends upon the course of exchange between England and India.

The following Table is used in keeping accounts in Bengali.

4 Cowries	make	1 Ganda.
5 Gandas	...	1 Buri.
20 Gandas	...	1 Pan.
4 Pans	...	1 Chauk.
4 Chauks	...	1 Kahan.

1 Cowri = 3 Krants = 4 Kags = 7 Dips = 8 Bats = 9 Dantis.

None of these denominations is the name of any actual coin; but the kahan stands for 1 rupee, the chauk for 4 annas, and the pan for 1 anna.

The notation used in Bengali accounts is peculiar. The kahans are represented by the ordinary numerals, 1 *chauk* by the vertical stroke ৷০, 2 *chauks* and 3 *chauks* by ৷৷০, and ৸০, 1 *pan* by the oblique stroke ৴০ and 2 *pans* and 3 *pans* by ৵০ and ৶০. Generally, in Bengali, if units of any denomination are represented by the ordinary numerals, their quarters are represented by ৷০, ৷৷০ and ৸০ and the quarters of these quarters by ৴০, ৵০ and ৶০.

Besides the theoretical cowries in the above Table, there are cowries or shells in actual use in the Bengal bazar for the payment of very small amounts, 1 pice being equal to 80 cowries generally.

In Behar and the North Western Provinces, the following subdivisions of the pice are in use:—

2 Damris	...make	1 Chhidam.
2 Chhidams	...	1 Adhela.
2 Adhelas	...	1 Paisa or Pice.

Table of Bengal Gold and Silver Weight.

160. In this Table,

4 Punkos	make	1 Dhán.
4 Dháns	...	1 Ratti.
6 Rattis	...	1 Anna.
96 Rattis or 12 Mashas or 16 Annas	...	1 Tola.

1 Tola = 180 grs. Troy.
32 Tolas = 1 lb. Troy.

Table of Bengal Doctors' Weight.

161. In this Table,

4 Jabs	make	1 Ratti.
5 Rattis	...	1 Dhán.
2 Dháns	...	1 Máshá.
2 Máshás	...	1 Tola.

1 Tola = 180 grs. Troy.

TABLE OF BRITISH INDIAN BAZAR WEIGHT.

162. In this Table used in weighing coarse articles,

5	Sikis or quarter rupees (in weight)	make	1 Kancha.
4	Kanchas or 5 rupees in weight	...	1 Chatak.
4	Chataks	...	1 Powa.
4	Powas	...	1 Seer.
5	Seers	...	1 Pusury.
8	Pusuries or 40 Seers	...	1 Maund.

1 Bazar Maund = 100 lbs. Troy = $82\frac{2}{7}$ lbs. Avoirdupois = $\frac{54}{49}$ of 1 Factory Maund = 1 Maund $4\frac{4}{49}$ Seers of Factory weight.

1 Seer = 80 rupees' weight = 80×180 or 14400 grs. Troy = $2\frac{1}{2}$ lbs. Troy = $\frac{72}{35}$ lbs. Avoir.

1 Factory Maund = $\frac{2}{3}$ cwt.

Bengal Regulation VII of 1833 (now repealed) first introduced the Tola of 180 grs. Troy as the standard unit of weight. The Table of weights given in that Regulation runs thus:—

8 Rattis	= 1 Masha	= 15	grs. Troy.
12 Masha	= 1 Tola	= 180	grs. Troy.
80 Tolas	= 1 Seer	= $2\frac{1}{2}$	lbs. Troy.
40 Seers	= 1 Maund	= 100	lbs. Troy.

The following is the Table of Bombay local weights :—

4	Dhans	make	1 Raktika.
8	Raktikas	...	1 Máshá.
4	Máshás	...	1 Tank.
72	Tanks	...	1 Seer.
40	Seers	...	1 Maund = 28 lbs. Avoir.
20	Maunds	...	1 Khandi.

The following is the Table of Madras local weights :—

10	Pagodas	make	1 Palam.
8	Palams	... ...	1 Seer = 10 oz. Avoir.
5	Seers	... ...	1 Bis.
8	Bis or 40 Seers	... ...	1 Maund = 25 lbs. Avoir.
20	Maunds	... ...	1 Khandi.

Table of Linear Measure.

163. In Bengal,

3 Jabs	make	1 Anguli.
4 Angulis	...	1 Muti.
3 Mutis	...	1 Bighat.
2 Bighats	...	1 Hát.
4 Háts	...	1 Dhanu.
1000 Dhanus	...	1 Kros or Kos.
2 Kroses	...	1 Gavyuti.
2 Gavyutis	...	1 Jojan.

4 Háts = 1 Káthá.
20 Káthás or 80 Háts = 1 Bigha or Rasi.

In the above Table 1 Kros = 1 mile 1 furlong 3 poles $3\frac{1}{2}$ yards English.

But generally, 1 Kros = 100 Rasis = 8000 Hats = 2 miles 2 furlongs 7 poles $1\frac{1}{2}$ yds. English.

1 Hat = 18 inches generally.

In Bombay, the half hath is called Vent; and the measuring rod or káthi for land is 9·4 ft.

In the North Western Provinces,

1 Ilahi guj = 33 inches.
3 Ilahi guj = 1 Bans.
20 Bans = 1 Jarib.

Table of Cloth Measure.

164. In Bengal,

3 Jabs	make	1 Anguli.
3 Angulis	...	1 Girah.
8 Girahs	...	1 Hát.
2 Háts	...	1 Guj = 1 yard.

In Bombay,

2 Angulis = 1 Tasu.
24 Tasus = 1 Guj = 27 in.

In Madras,

1 Kovid = 18·6 in.

Table of Land Measure.

165. In Bengal,

1	Square Cubit makes ...	1 Ganda.
20	Gandas or 5 Cubits long by 4 wide make	1 Chatak.
16	Chataks or 80 Cubits long by 4 wide ...	1 Katha.
20	Kathas or 80 Cubits long by 80 wide ...	1 Bigha.

A Katha $= 80 \times 4$ or 320 sq. cubits.
$= 120 \times 6$ or 720 sq. ft.

A Bigha $= 80 \times 80$ or 6400 sq. cubits.
$= 120 \times 120$ or 14400 sq. ft.
$= \frac{14400}{9}$ or 1600 sq. yds.

From Art. 148,

1 acre $= 4840$ sq. yds.
$= 3 \times 1600$ sq. yds. $+ 40$ sq. yds.
$= 3$ bighas $+ \frac{1}{40}$ bigha
$= 3\frac{1}{40}$ bighas.

The Benares or Ghazipur Bigha.
$= 3600$ sq. Benares guj $= 3136$ sq. yds.
$= 2$ Bengal bighas nearly.

In the North-Western Provinces,

20 Biswa $= 1$ bigha.
1 Bigha $= 3600$ sq. guj. Ilahi.
$= 3025$ sq. yds.

In Bombay,

$34\frac{1}{36}$	Square Hats make	1 Kathi.
20	Katiyo ...	1 Pand.
20	Pands ...	1 Bigha $= 3897 \cdot 4$ sq. yds.
6	Bighas ...	1 Rukeh.
120	Bighas ...	1 Chahur.

In Madras,

1 Manai $=$ 2400 sq. ft.
24 Manai $=$ 1 Kani $= 6400$ sq. yds.

Table of Bengal Lime Measure.

166. 2 ft. 3 in. long by 1 ft. 8 in. wide by 9 in. deep $=$ 1 Ferah weighing $1\frac{1}{4}$ mds.
80 Ferahs or 225 cub. ft. $=$ 100 maunds.

Table of Bengal Liquid Measure.

167.	4 Chataks	make	1 Powa.
	4 Powas	...	1 Seer.
	40 Seers	...	1 Maund.

Table of Corn Measure.

168. In Bengal,

5 Chataks	make	1 Kunki.
4 Kunkis	...	1 Rek.
4 Reks	...	1 Pally.
20 Pallies	...	1 Solly.
16 Sollies	...	1 Kahun = 40 mds.

In Bombay,

36 Tanks	make	1 Tipari.
2 Tiparis	...	1 Ser.
4 Sers	...	1 Payali = $2\frac{4}{5}$ lb. Avoir.
16 Payalis	...	1 Pharo.
8 Phara	...	1 Khandi.
25 Phara	...	1 Muda.

$8\frac{1}{2}$ Payalis of lime = 1 Pharo.
$17\frac{1}{4}$ Payalis of rice = 1 Pharo.

In Madras,

8 Ollaks	make	1 Padi.
8 Padis	...	1 Markal = 750 cub. in.
5 Markals	...	1 Phara.

Table of Bengal Time.

169.	60 Anupals	make	1 Bipal.
	60 Bipals	...	1 Pal.
	60 Pals	...	1 Ghari or Danda = 24′
	$7\frac{1}{2}$ Gharis	...	1 Prahar = 3 hrs.
	8 Prahars	...	1 Day.
	7 Days	...	1 Hafta.
	15 Days	...	1 Paksha.
	2 Pakshas	...	1 Mas.
	2 Mas	...	1 Ritu.
	6 Ritus	...	1 Vatsar.
	12 Vatsars	...	1 Yug.

The Hindoo year of the Sakabda era begins from the 1st of Baisakh, corresponding with about the 11th, 12th, or 13th of April. Its length = 365 days 15 Dandas 31 Pals 31 Bipals and 24 Anupals. The Samvat year begins from the first day after the new moon in Chaitra. The two Hindu eras in common use are the Samvat, which commenced from 57 B. C., and the Sakabda, which commenced from 78 A. D.

The following are the names of the Hindu months, with their astronomical lengths in days, hours, and minutes.

Months.		Days.	hrs.	min.
Baisakh	...	30	22	12·8.
Jyaistha	...	31	9	40·8.
Ashadh	...	31	14	39·2.
Sravan	...	31	11	16·8.
Bhadra	...	31	0	52·0.
Asvin	...	30	10	56·8.
Kartic	...	29	21	38·8.
Agrahayan	...	29	12	9·6.
Paus	...	29	8	21·2.
Magh	...	29	10	54·4.
Falgun	...	29	19	21·6.
Chaitra	...	30	8	8·4.

In ordinary reckoning, a fraction of a day belonging to a month is generally counted as one entire day, the difference between such fraction and a whole day being deducted from the length of the next month.

170. Act XXXI of 1871, which provides for the adoption of a uniform system of Weights and Measures of Capacity, enacts amongst other things, that the unit of weight shall be the Ser equal in weight to the French Kilogramme = 2 lbs. 3oz. 4·3830 drams Avoirdupois, and the unit of the measure of capacity, the measure containing one such Ser of water at its maximum density weighed in a vacuum.

SECTION II. NOTATION OF CONCRETE INTEGERS. REDUCTION.

171. The Notation of concrete integers is extremely simple. We have only to write down the numbers of the successive denominations one after another, with the abbreviation for the denomination of each number, above or by the side of the same.

Thus, *five pounds six shillings and eight pence* will be written

$$\begin{array}{l} \qquad\qquad\quad \text{£. } s. \; d. \\ \text{as £5. } 6s. \; 8d. \text{ or } 5. \; 6. \; 8. \end{array}$$

172. Def. Reduction is the method of expressing a number of one denomination in units of another.

173. Rule I. To reduce a quantity from higher denominations to a lower one, multiply the number of the highest denomination in the given quantity by the number connecting that denomination with the next lower, and to the product add the number of the next lower denomination, if any, in the given quantity; repeat this process for each succeeding denomination till the one required is arrived at.

Rule II. To reduce a quantity from a lower denomination to a higher, divide the given quantity by the number connecting its denomination with the next higher, and set down the remainder, if any, under its own denomination, and the quotient under the next higher; repeat this process with the first and the successive quotients till the required denomination is arrived at.

Ex. 1. Reduce 28 Rs. 13 a. 3 pice to pies.

By the Rule we have

$$\begin{array}{rl}
28 & \text{Rs.} \quad 13 \text{ a.} \quad 3 \text{ pice.} \\
16 & \\ \hline
168 & \\
28 & \\ \hline
448 & \text{a.} \\
13 & \text{a.} \\ \hline
461 & \text{a.} \\
4 & \\ \hline
1844 & \text{pice.} \\
3 & \;\;" \\ \hline
1847 & \;\;" \\
3 & \\ \hline
5541 & \text{pies.}
\end{array}$$

Ex. 2. Reduce 1000 tolas to seers, &c.

By the Rule we have

$$\begin{array}{l|l} 5 & 1000 \text{ tolas.} \\ \hline 4 & \quad 200 \text{ cht.} \\ \hline 4 & \quad\ \ 50 \text{ pow.} \\ \hline & \quad\ \ 12 \text{ seers. } 2 \text{ pows.} \end{array}$$

The *reason for the Rules* will be seen below.

In Ex. 1,

$$\begin{aligned} 28 \text{ Rs. } 13 \text{ a. } 3 \text{ pice} &= 28 \times 16 \text{ a. } + 13 \text{ a.} + 3 \text{ pice} \\ &= 461 \text{ a.} + 3 \text{ pice} = 461 \times 4 \text{ pice} + 3 \text{ pice} \\ &= 1847 \text{ pice} = 1847 \times 3 \text{ pies} \\ &= 5541 \text{ pies.} \end{aligned}$$

In Ex. 2,

$$\begin{aligned} 1000 \text{ tolas} &= \frac{1000}{5} \text{ cht.} = 200 \text{ cht:} \\ &= \frac{200}{4} \text{ pows} = 50 \text{ pows} \\ &= \frac{50}{4} \text{ seers} = 12 \text{ seers } 2 \text{ pows.} \end{aligned}$$

174. Here may be noticed the advantages of the use of the Metric System.

In this system, Reduction from one denomination to another is performed by a simple transposition of the decimal point.

Thus, 3 kilom. 5 hectom. 6 dekam. 7 m. 9 decim.

$$\begin{aligned} &= (3 \times 1000 + 5 \times 100 + 6 \times 10 + 7 + \tfrac{9}{10})\text{m.} \\ &= 3567{\cdot}9 \text{ m.} \\ &= 356{\cdot}79 \text{ dekam.} \\ &= 35{\cdot}679 \text{ hectom.} \\ &\quad \&c. \end{aligned}$$

Ex. XXIII.

1. Reduce

(1) 12 Rs. 13 a. 4 p. to pies; and 1000 pies to rupees.
(2) 23 Rs. 14 a. 5 p. to pies; and 250 annas to rupees.
(3) 159 Rs. 0 a. 7 p. to pies; and 500 annas to rupees.
(4) 234 Rs. 15 a. to annas; and 196 annas to rupees.
(5) £ 21. 2*s*. 3*d*. to pence; and 500*d*. to pounds.

(6) £ 32. 3*s*. $4\frac{1}{2}$ *d*. to farthings; and 100*d*. to pounds.

(7) 225 half-sovereigns to pence; and 5633 pence to half-guineas.

(8) 529 half-crowns to pounds; and £52. 5*s*. to crowns.

(9) 1 ℔. 2 oz. 3 dwts to grains; and 1234 grs. to ounces.

(10) 12 ℔s. 13 oz. 14 dwts. to pennyweights; and 123414 grs. to pounds.

(11) 3 tons. 14 cwt. 3 qrs. to pounds; and 1000 oz. to pounds.

(12) 5 tons. 15 cwt. 16 lbs. to grains Troy; and 10000 pounds to tons.

(13) 25 tolas 12 rattis to rattis; and 1000 rattis to tolas.

(14) 31 maunds 32 seers 3 chts. to kanchas; and 1000 tolas to seers.

(15) 100 maunds to tolas; and 10000 tolas to maunds.

(16) 1 mile 7 fur. 13 po. to yards; and 2000 yards to miles.

(17) 23 yds. 2 ft. 7 in. to inches; and 525 in. to yards.

(18) 1 ac. 2 ro. 3 sq. po. to square yds; and 3000 sq. yds. to acres.

(19) 15 cub. ft. 27 cub. in. to cubic inches; and 500 cub. ft. to cub. yds.

(20) 2 kroses 50 dhanus to cubits; and 15678 cubits to kroses.

(21) 5 kroses 20 rasis to cubits; and 1000 rasis to kroses.

(22) 28 bighas 15 kathas to kathas; and 525 káthás to bighas.

(23) 39 bighas 16 káthás to káthás; and 1650 chataks to bighas.

(24) 1 kilog. 2 hectog. 3 dekag. to grams; and 987654 millig. to grams.

(25) 4 kilom. 5 hectom. 6 dekam. to centimetres; and 78910 centim. to dekametres.

2. How many pence are there in £100, and how many pounds in 1600 pence?

3. How many acres are there in 640 sq. miles, and how many bighas in 144 káthás?

4. How many minutes are there in 1 Julian year, and how many dandas in 1 week?

SECTION III. COMPOUND ADDITION.

175. DEF. The Addition of concrete numbers of different denominations is called COMPOUND ADDITION.

It is evident that the numbers to be added together must be of the same *kind.* For otherwise, the Addition would be meaningless and absurd. Thus there can be no meaning in saying that 5 *rupees* are to be added to 6 *seers*, or the like.

176. RULE. Arrange the numbers so that units of the same denomination may be in the same column. Add up the column of the lowest denomination; divide the sum by the number of units of its denomination contained in one unit of the next higher denomination, set down the remainder under the column added, and carry the quotient to be added to the column of the next higher denomination. Repeat this process for every column.

If a column of any intermediate denomination be wholly wanting, indicate its existence by zeros.

Ex. 1. Add together. £3. 13*s.* $4\frac{1}{2}$*d.* £5. 12*s.* $6\frac{3}{4}$*d.* and £8. 10*s.* $8\frac{1}{4}$*d.*

By the Rule we have

£	*s.*	*d.*
3	13	$4\frac{1}{2}$
5	12	$6\frac{3}{4}$
8	10	$8\frac{1}{4}$
17	16	$7\frac{1}{2}$

Reason for the Rule.

$(2+3+1)$ farthings $=6$ farthings $=1$*d.* $+$ 2 farthings; so we put down 2 farthings or $\frac{1}{2}$ *d.* and carry 1*d.* to be added to $(4+6+8)$*d.* thus getting in all 19*d.* which $=$ 1*s.* 7*d.*; so we put down 7*d.*, and carry 1*s.* to be added to $(13+12+10)$*s.* thus getting in all 36*s.* or £ 1. 16*s.*; we then put down 16*s.*,

and carry £1 which added to £(3+5+8) makes in all £17. The required sum $\therefore$ = £17. 16*s*. $7\frac{1}{2}$*d*.

Ex. 2. Add together 2 mds. 3 seers 3 chts., 5 mds. 4 seers 2 chts., 17 mds. 3 chts., 5 pus. 2 seers. 2 chts., and 3 mds. 7 pus. 3 chts.

By the Rule we have

mds.	pus.	seers	pow.	chts.
2	0	3	0	3
5	0	4	0	2
17	0	0	0	3
0	5	2	0	2
3	7	0	0	3
28	5	4	3	1

Ex. XXIV.

1. Add together

(1) £	s.	d.	(2) £	s.	d.	(3) £	s.	d.
1	2	3	20	19	11	39	15	$7\frac{1}{4}$
4	5	$6\frac{1}{2}$	21	18	$10\frac{1}{2}$	38	14	6
7	8	$9\frac{3}{4}$	22	17	9	37	13	$5\frac{3}{4}$
10	11	$0\frac{1}{4}$	23	16	$8\frac{3}{4}$	36	12	4

(4) Rs.	a.	p.	(5) Rs.	a.	p.	(6) Rs.	a.	p.
35	11	3	43	7	1	57	15	7
34	10	2	44	6	2	69	13	5
33	9	1	45	5	3	72	9	11
32	8	0	46	4	4	88	11	9

(7) tons.	cwt.	qrs.	lbs.	(8) tons.	cwt.	qrs.	lbs.
20	15	3	16	13	14	1	15
31	16	2	17	12	13	2	14
42	17	1	18	11	12	3	13
53	18	0	19	10	11	0	12

(9)

mds.	seers.	chts.
32	33	9
33	34	10
34	35	11
35	36	13

(10)

mds.	seers.	chts.
45	23	3
47	21	5
53	25	7
63	35	9

(11)

yds.	ft.	in.
3	2	7
7	1	8
9	2	11
11	0	5

(12)

miles.	fur.	pol.
68	5	6
57	4	8
46	3	9
35	2	10

(13)

£.	s.	d.
5	7	9
11	13	$4\frac{1}{2}$
23	15	7
28	19	11

(14)

£	s.	d.
21	13	5
25	15	6
27	9	$1\frac{1}{2}$
28	14	$3\frac{3}{4}$

(15)

£.	s	d.
19	7	9
47	13	$5\frac{1}{2}$
68	9	$4\frac{1}{4}$
73	16	6

(16)

Rs.	a.	p.
55	13	3
63	14	6
75	12	9
64	14	8

(17)

Rs.	a.	p.
79	11	6
8	12	3
9	15	9
11	7	3

(18)

Rs.	a.	p.
18	14	8
23	9	4
25	10	6
73	11	8

2. Find the sum of 15 bighas 16 kathas 14 chataks; 16 bighas 17 kathas 13 chataks; and 17 bighas 18 kathas 12 chataks.

3. Add together 26 sq. yds. 7 sq. ft. 9 sq. in.; 25 sq. yds. 8 sq. ft. 10 sq. in.; and 24 sq. yds. 9 sq. ft. 11 sq. in.

4. Find the sum of 3 lbs. 4 oz. 5 dwts.; 4 lbs. 5 oz. 6 dwts.; and 19 lbs. 7 oz. 13 dwts.

5. Add together 16 Rs. 15 a. 11 p.; 28 Rs. 13 a.; 56 Rs. 10 a. 9 p.; and 106 Rs. 5 a.

6. Add together 7 miles 5 fur. 6 po.; 9 miles 1 fur. 28 po.; and 91 miles 3 fur. 37 po.

SECTION IV. COMPOUND SUBTRACTION.

177. DEF. COMPOUND SUBTRACTION is the method of finding the difference between two concrete numbers of different denominations.

As in Compound Addition, it is evident that the numbers between which the difference is to be found out must be of the same *kind*.

178. RULE. Place the less number below the greater so that numbers of the same denomination may be in the same column. Subtract the number of each denomination in the subtrahend from the corresponding number in the minuend, and put down the difference below. If a number of any denomination in the minuend is less than the corresponding number in the subtrahend, increase it by the number of units of its own denomination contained in one unit of the next higher, and then perform the subtraction, taking care to add 1 to the number of the next higher denomination in the subtrahend.

The several partial differences taken together will be the difference required.

Ex. Subtract 5 Rs. 11 a. 3 pice from 18 Rs. 13 a. 2 pice.

By the Rule we have

Rs.	a.	p.
18	13	2
5	11	3
13	1	3

Reason for the Rule.

To subtract one concrete number from another is to subtract the number of each denomination in the subtrahend from the corresponding number in the minuend.

Now, as 3 pice cannot be subtracted from 2 pice, we increase the latter by 1 anna or 4 pice making it $4+2$ or 6 pice, and then $(6-3)$ pice $=3$ pice; so we put down 3 in the column of pice; and as we have added 1 anna to the minuend, to keep the difference unchanged, we add 1 anna to the 11 annas in the subtrahend (Art. 34. Prop. I) thus making it 12 annas; and

13 annas − 12 annas = 1 anna, which we put down in the annas' column; and lastly 18 Rs. − 5 Rs. = 13 Rs. Thus the difference required = 13 Rs. 1 a. 3 pice.

Ex. XXV.

1. Subtract

(1) £15. 15*s.* 6*d.* from £26. 14*s.* $3\frac{3}{4}$*d.*
(2) £23. 19*s.* 11*d.* from £70. 13*s.* 9*d.*
(3) £36. 5*s.* $7\frac{1}{2}$*d.* from £84. 17*s.* 11*d.*
(4) £234. 9*s.* 5*d.* from £500. 0*s.* 6*d.*
(5) 7 Rs. 8 a. 9 p. from 10 Rs. 9 a. 8 p.
(6) 70 Rs. 7 a. 8 p. from 88 Rs. 6 a. 5 p.
(7) 56 Rs. 13 a. 11 p. from 96 Rs. 7 a. 8 p.
(8) 37 Rs. 12 a. 9 p. from 59 Rs. 13 a. 3 p.
(9) 3 lbs. 11 oz. 12 dwts. from 28 lbs. 8 oz. 10 dwts.
(10) 15 lbs. 0 oz. 15 drs. from 17 cwt. 3 qrs. 10 lbs.
(11) 1 md. 19 seers 7 chts. from 15 mds. 10 seers 3 chts.
(12) 17 mds. 28 seers 13 chts. from 29 mds. 8 seers.
(13) 1 mile 2 fur. 3 po. from 69 miles 5 fur. 37 po.
(14) 2 kroses 15 rasis from 5 kroses 10 rasis.
(15) 3 sq. yds. 7 sq. ft. 56 sq. in. from 20 sq. yds. 6 sq. ft. 20 sq. in.
(16) 15 cub. ft. 16 cub. in. from 24 cub. ft. 9 cub. in.
(17) 3 bighas 10 kathas 8 chts. from 12 bighas 8 kathas 6 chts.
(18) 7 kilog. 8 dekag. 9 gram. from 9 kilog. 8 hectog. 7 dekag.

2. Find the difference between

(1) 7 weeks 5 days 10 hrs. and 15 weeks 3 days 5 hrs.
(2) 6 prahars 2 dandas 56 pals and 7 prahars 1 danda 10 pals.
(3) 6 hrs. 7′ 8″ and 22 hrs. 2′ 5″.
(4) 5° 10′ 15″ and 18° 6′ 10″.
(5) 10 annas 13 gandas 1 cowry 1 krant and 5 annas 5 gandas 2 cowries 2 krants.
(6) 5 a. 6 gan. 2 cowr. 2 kr. and 2 a. 13 gan. 1 cowr. 1 kr.

SECTION V. COMPOUND MULTIPLICATION.

179. DEF. The Multiplication of a concrete number of different denominations by an abstract number is called COMPOUND MULTIPLICATION.

As we have already seen (Art. 38) the multiplier must always be an abstract number. There are *apparent* exceptions to this rule, but they are *not real* exceptions. They will be noticed in Arts. 184 and 198.

180. RULE. Place the multiplier below the lowest denomination of the multiplicand; multiply the number of the lowest denomination by the multiplier; divide the product by the number of units of its own denomination contained in one unit of the next higher, put down the remainder, if any, under the denomination in question, and carry the quotient to be added to the product of the multiplier by the number of the next higher denomination in the multiplicand. Repeat this process for each of the given denominations.

Ex. 1. Multiply 16 Rs. 15 a. 2 p. by 7.

By the Rule we have

Rs.	a.	p.
16	15	2
		7
118	10	2

Reason for the Rule

To multiply a concrete number of several denominations is to multiply the number of each separate denomination by the multiplier, and to take the sum total of all the partial products, after performing the necessary reductions.

Thus, 2 pies $\times 7 = 14$ pies $= 1$ anna $+ 2$ pies; so we put down 2 pies in the pies' column. Next 15 a. $\times 7 = 105$ annas, and to this must be added the 1 anna already obtained, and we thus get $105 + 1$ or 106 annas $= 6$ Rs. 10 a., and we put down 10 annas in the column of annas. Lastly we multiply 16 Rs. by 7 and get 112 Rs. to which must be added 6 Rs., thus giving us 118 Rs, Thus the product is 118 Rs. 10 a. 2 pies.

Ex. XXVI.

1. Multiply
   (1) £15. 10*s*. 6*d*. by 2, 3, and 5.
   (2) £16. 11*s*. 7½*d*. by 4, 5, and 6.
   (3) £17. 17*s*. 9¾*d*. by 7, 8, and 9.
   (4) £28. 12*s*. 8*d*. by 10, 12, and 15.
   (5) 20 Rs. 12 a. 9 p. by 8, 12, and 16.
   (6) 25 Rs. 15 a. 3 p. by 5, 6, and 8.
   (7) 32 Rs. 8 a. 6 p. by 4, 6, and 12.
   (8) 56 Rs. 14 a. 3 p. by 3, 6, and 9.
   (9) 10 lbs. 6 drams. 2 scr. 15 grs. by 4, 5, and 10.
   (10) 19 cwt. 2 qrs. 14 lbs. by 4, 8, and 12.
   (11) 5 mds. 10 seers 3 powas by 6, 8, and 9.
   (12) 17 mds. 15 seers 15 chts. by 15, 20 and 32.
   (13) 2 wks. 5 days 15 hrs. 10′ by 15, 20, and 32.
   (14) 5 days 21 hrs. 15′ by 31, 32, and 40.
   (15) 6 kilog. 5 hectog. 6 grams 7 centig. by 5, 10, and 15.
   (16) 8 dekam. 7 metres 6 decim. by 2, 4, and 8.

2. Find the product of
   (1) £15. 10*s*. 10*d*. by 3 and 30.
   (2) 18 Rs. 10 a. 6 p. by 10 and 100.
   (3) 15 cwt. 2 qrs. 24 lbs. by 50 and 100.
   (4) 16 lbs. 15 dwts. 14 grs. by 20 and 200.
   (5) 5 mds. 6 seers 7 chts. by 30 and 300.
   (6) 30 mds. 20 seers 8 chts. by 10 and 15.

SECTION VI. COMPOUND DIVISION.

181. **Def.** The Division of a concrete number of different denominations by an abstract number or another concrete number is called COMPOUND DIVISION.

As we have already seen (Art. 51) the quotient in the former case will be a concrete number indicating the magnitude of each part after division, and in the latter, it will be an abstract number indicating the number of times that the divisor is contained in the dividend.

182. When the divisor is an abstract number proceed thus:

RULE. Place the numbers as in Simple Division. Divide the number of the highest denomination in the dividend by the divisor; the quotient will be of the same denomination; reduce the remainder, if any, to the next lower denomination, add to it the number of the corresponding denomination in the dividend, and then divide the sum by the divisor. Repeat this process down to the lowest denomination. The several partial quotients taken together will be the quotient required.

EX. Divide 151 Rs. 3 a. 2 pies by 25.

```
          Rs.   a.   p.
    25)   151   3    2 (6 Rs.
          150
          ---
            1
           16
          ---
           16
            3
          ---
      25)  19 ( 0 a.
           12
          ---
          228
            2
          ---
     25 ) 230 ( 9 5/25 pies
          225
          ---
            5
```

∴ the quotient reqd. is 6 Rs. 0 a. $9\frac{5}{25}$ pies.

Reason for the Rule.

The number of rupees to be divided is 151, and these rupees divided into 25 parts give 6 rupees for each part, with 1 R. over; and 1 R. = 16 annas; which together with the 3 a. in the dividend give 19 a. as the total number of annas to be divided; as 19 a. cannot be divided into 25 parts, we reduce these to pies, getting 19×12 or 228 pies which together with the 2 pies in the dividend give

230 pies as the total number of pies to be divided, and 230 pies $\div$ 25 give $9\frac{5}{25}$ pies for each part; and nothing more remains to be divided. Thus the quotient is 6 Rs. $9\frac{5}{25}$ pies.

183. When the dividend and the divisor are both concrete numbers, proceed thus:

RULE. Reduce both the dividend and the divisor to the same denomination, and then proceed as in Simple Division.

Ex. Divide £ 57. 13*s.* by £ 5. 10*s.* 6*d.*

By the Rule we have

$$\begin{aligned} £\,57.\ 13s. &= (57 \times 20 + 13)s. &&= 1153s. \\ &= (1153 \times 12)d. &&= 13836d. \\ £\,5.\ 10s.\ 6d. &= 110s. + 6d. &&= (110 \times 12 + 6)d. \\ & &&= 1326d. \end{aligned}$$

$$\begin{array}{r} 1326)13836(10 \\ \underline{1326} \\ 576 \end{array}$$

$\therefore 10\frac{576}{1326}$ is the quotient reqd.

Reason for the Rule.

To divide £ 57. 13*s.* by £ 5. 10*s.* 6*d.* is to find how often the latter is contained in the former, *i. e.*, how often the amount 1326*d.* is contained in 13836 *d.*, *i. e.*, how often the number 1326 is contained in 13836; so that we have only to divide 13836 by 1326 as in Simple Division.

Ex. XXVII.

1. Divide—

(1) £ 9. 10*s.* 5*d.* by 2, 3, and 4.
(2) £ 55. 15*s.* 6*d.* by 5, 6, and 7.
(3) £ 67. 17*s.* 7*d.* by 8, 9, and 10.
(4) £ 226. 13*s.* 4*d.* by 72, 73, and 75.
(5) 56 Rs. 14a. 3p. by 10, 12, and 14.
(6) 150 Rs. 12a. 8p. by 15, 16, and 18.
(7) 225 Rs. 10a. 10p. by 19 and 31.
(8) 640 Rs. 8a. 6p. by 8 and 18.
(9) 56 gals. 3 qts. by 4 and 14.
(10) 72 qrs. 2 bus. 2 pks. by 8 and 16.
(11) 17 cwt. 2 qrs. 14lbs. by 9 and 19.
(12) 1 ton. 18 cwt. 16 lbs. by 15 and 20.

2. Divide–

(1) £ 15. 10*s*. 9*d*. by £ 1. 5*s*. 6*d*.
(2) £ 28. 9*s*. by £ 3. 3*s*. 3*d*.
(3) 52 Rs. 11 a. by 3 Rs. 7 a. 7 p.
(4) 16 Rs. 15 a. 11 p. by 4 Rs. 3 a. 2 p.
(5) 22 sq.miles by 3 sq.yds. 5 sq.ft.
(6) 22 sq.yds. 8 sq.ft. by 5 sq.yds. 2 sq.ft.
(7) 11 bighas 10 kathas by 2 bighas 11 kathas.
(8) 12 bighas 6 kathas by 4 bighas 2 kathas.
(9) 22 hrs. 55′ by 3 hrs. 10′.
(10) 16 hrs. 40′ by 3 hrs. 15′.
(11) 14 dwts. 10 grs. by 3 dwts. 12 grs.
(12) 42 mds. 10 seers by 8 seers 8 chts.

MISCELLANEOUS QUESTIONS AND EXAMPLES.

184. In working out Examples, the remarks in Art. 77 should be borne in mind.

Ex. 1. A person gives 5 Rs. 4 a. to each of 13 men; how much does he spend altogether?

Here, the total amount spent
= (amount given to one man) × (the number of men)
= (5 Rs. 4 a.) × 13
= 68 Rs. 4 a.

In the above process, it may appear at first sight that the multiplier 13 is a concrete number being 13 men; but in fact it is not so, for we have taken 5 Rs. 4 a. 13 times simply, and not evidently 13 men times, which would be absurd.

Ex. 2. A person by giving 5 Rs. 4 a. to each man, spends 68 Rs. 4 a.; how many men were there?

Here the number of men reqd.
= the number of times that the sum of 5 Rs. 4 a. is contained in 68 Rs. 4 a.
= 68 Rs. 4 a. ÷ 5 Rs. 4 a.
= 1092 a. ÷ 84 a. = 13.

Here it may appear at first sight that in dividing one concrete number 68 Rs. 4 a. by another concrete number 5 Rs. 4 a. we get a concrete number, *viz.*, 13 men for the quotient; but in fact that is not so. The quotient we get is not 13 men, but

the abstract number 13, and it so happens only from the nature of the question that the *number* of men required to be found out is equal to this quotient 13, and has to be ascertained by means of the operation of Division indicated above.

Ex. 3. How many rupees, half-rupees, quarter-rupees, and two-anna pieces are there in 140 Rs. 10 a., supposing there to be an equal number of each ?

Here, the number reqd. = the number of times that the sum of 1 rupee + 1 half-rupee + 1 quarter-rupee + 1 two-anna piece is contained in 140 Rs. 10 a.

Now 1 rupee + 1 half-rupee + 1 quarter-rupee + 1 two-anna piece = 15 two-anna pieces
and 140 Rs. 10 a. = 2250 a.
= 1125 two-anna pieces ;
∴ the number reqd. = the number of times that 15 two-anna pieces are contained in 1125 two-anna pieces

$$=\frac{1125}{15}=75.$$

Ex. 4. A bag contains a certain number of sovereigns, twice as many half-sovereigns and five times as many crowns ; and the whole sum in the bag amounts to £653. 5*s*. Find the number of coins of each kind.

Here, if we divide the coins in the bag into groups of 1 sovereign, 2 half-sovereigns and 5 crowns each, the number of such groups will be the same as the number of sovereigns contained.

Hence the no. of sovereigns in the bag
= the no. of times that one of these groups is contained in the given sum
= £. 653. 5*s*. ÷ (1 sovereign + 2 half-sovereigns + 5 crowns) = 13065*s*. ÷ 65*s*.

$$=\frac{13065}{65}=201.$$

And the no. of half sovereigns = 2 × 201 = 402,
and......................crowns = 5 × 201 = 1005.

Ex. 5. A milkman buys milk for 8 Rs. at 7 seers a rupee, and sells it adulterated with water at 8 seers a rupee, making

a profit of 2 pice in the rupee; how much water has he added? and how much more water would have raised his whole profit to 1 rupee?

Here the quantity of milk bought $= 8 \times 7$ or 56 seers.

Now, in first case, there is a profit of 2 pice in the rupee or 8×2 pice or 4 annas in all; and so there has been milk sold for 8 Rs. 4 a. or $8\frac{1}{4}$ Rs. at 8 seers a rupee;

$\therefore$ the quantity of milk sold $= (8 \times 8\frac{1}{4})$ seers $= 66$ seers.
and.................water added $= (66 - 56)$ seers $= 10$ seers.

Similarly, in the second case,
the quantity of milk sold $= (8 \times 9)$ seers $= 72$ seers;
$\therefore$ ....water added $= (72 - 56)$ seers $= 16$ seers, and
$\therefore$ the additional quantity of water required in the second case $= (16 - 10)$ seers $= 6$ seers.

Ex. 6. A traveller who walks 20 miles a day, is followed in the same route by another traveller who walks 25 miles a day, and starts 3 days later. When will the latter overtake the former; and what will be the distance between them 10 days after the latter starts?

Here, the first traveller is 3×20 or 60 miles in advance of the second; and $\therefore$ before he is overtaken, the latter must walk 60 miles more than the former.

But he walks daily $(25 - 20)$ or 5 miles more than the first traveller;

$\therefore$ he will overtake the first traveller in $\frac{60}{5}$ or 12 days.
Again, on the 10th day after starting, the second traveller is (10×25) miles or 250 miles from the starting point; and the first $(10 \times 20 + 60)$ miles or 260 miles from the same point;
$\therefore$ the distance between them $= (260 - 250)$ miles $= 10$ miles.

Ex. 7. A man buys 3 yds. of woollen cloth, 5 yds. of silk, and 7 yds. of linen, for 23 Rs. 10 a. A yard of silk is worth 4 yards of linen, and a yard of woollen cloth is worth 3 yards of silk. What is the price of each per yard?

By the question,
5 yds. of silk are worth (4×5) yds. or 20 yds. of linen, and 3 yds. of woollen cloth worth (3×3) yds. of silk, *i. e.*, worth

$(4 \times 3 \times 3)$yds. or 36 yds. of linen; so that the man would have spent the same amount of money if instead of buying 3 yds. of woollen cloth and 5 yds. of silk, he had bought $(36 + 20)$ yds. of linen; so that $(36 + 20)$ yds. $+ 7$ yds. or 63 yds. of linen are worth 23 Rs. 10 a.

∴ the price of linen per yd. = 23 Rs. 10 a. ÷ 63

$= \frac{378}{63}$a. = 6a.

and..............silk..............= 4×6 a. = 1 R. 8a.

and..............woollen cloth... = 3×1 R. 8a. = 4 Rs. 8a.

Ex. XXVIII.

I.

1. Explain clearly how concrete quantities are numerically represented.

2. Give the different meanings of the term *carat.* What is the weight of a shilling, and how many shillings are equal to a rupee in weight?

3. How many drams of Apothecaries' weight are there in 1000 dwts? and how many drams Avoirdupois are there in the same?

* How many penny-weights are there in 1 cwt.?

4. What is Reduction?

Reduce 33 half-crowns to farthings, and 3300 pies to rupees.

5. How many barley-corns will reach round the Earth, supposing its circumference to be 25000 miles?

6. A room is 10 ft. 6 in. from the floor to the ceiling. How many copies of a book, 1in. thick, must be piled upon the floor, one above another, so as to reach to the ceiling?

II.

1. What are the different uses of Troy weight and Avoirdupois weight?

Supposing the revenue of India to be 60 crores of rupees, and to be collected in rupees, what would be its weight in maunds and pounds Troy; and how many carts would be required to carry it, supposing each cart to carry 16 maunds?

2. Supposing the population of India to be 200 millions, what would be the amount raised, if every individual in India were to contribute 1 cowry (there being 80 cowries in a pice)? How muchmore money would be raised if each individual were to contribute 1 pie?

3. If each individual on an average consume half a seer of salt per month, what would be the total quantity of salt consumed in 1 year in India, supposing its population to be 200 millions; and what would be its price, at 2 annas a seer?

4. What is the relation between a linear foot and a square foot?

Shew how to find the number of square inches in 1 square foot.

5. England including Wales contains 58660 square miles. What is the area of England and Wales in bighas?

6. What is the Julian year, and what correction has been made in it by Pope Gregory XIII?

How many hours are there in an average Julian year?

III.

1. What is Compound Addition? State the Rule for Compound Addition.

Find the value of 1 crown + 1 mark + 1 pound + 1 guinea + 1 moidore.

2. How many pice are there in 50 rupees, 50 half-rupees, and 50 quarter-rupees taken together?

3. A gentleman's eldest child was born on the 5th of August 1869, his second child on the 13th of April 1872, his third child on the 7th of October 1874, and his fourth child on the 23rd of March 1877. What will be the sum of the ages of all these children on the 30th of September 1878; and what was the day and year of the father's birth, supposing that he had lived 8000 days on the birth day of his eldest child?

4. A person goes to the bazar with 86 Rs. 4a. in his pocket, and spends 25 Rs. 4 a. in the purchase of books, 26 Rs. 8 a. in the purchase of cloth, and 4 Rs. 14 a. in the purchase of stationery. How much money has he still left?

5. A milkman supplies a customer with milk at the rate of 3 seers a day for the whole of the year 1876. What is the total quantity of milk supplied, and what is its price, if milk sells at 2 annas a seer?

6. Between the execution of Louis XVI, January 21 of 1793, and the battle of Waterloo, June 17 of 1815, how many days intervened?

IV.

1. In Compound Subtraction, when a number of any denomination in the minuend is less than the number of the same denomination in the subtrahend, how do you proceed, and what is your reason for proceeding in that way?

2. The first class railway fare from Howrah to Benares is 44 Rs. 8 a. 6 p., and from Howrah to Delhi is 89 Rs. 7 a. Find the difference between the two fares, and the distance between Benares and Delhi by railway, supposing the fares to be at the rate of 1 a. 6 p. per mile.

3. A bag contains a certain number of rupees, twice as many half-rupees, three times as many four-anna pieces, and four times as many two-anna pieces; and the whole sum in the bag is 81 Rs. 4 a. How many coins are there of each kind?

4. A grocer buys 7 mds. of sugar at 13 Rs. 4 a. per maund, and 9 mds. at 12 Rs. 8 a. per maund, and mixes the two quantities together. At what price per maund must he sell the mixture to secure a profit of 20 Rs. on the whole?

5. A box with its contents weighs 7 mds. 8 seers, and the weight of the box is 1 md. 15 seers. Find the weight of its contents.

6. What is the relation between a linear and a square bigha? The area of India is 1300000 square miles. How many bighas does it contain?

V.

1. What is Compound Multiplication? Can you multiply one concrete number by another?

Find the price of 58 mds. of rice at 4 Rs. 12 a. a maund.

2. A dealer buys 150 mds. of rice at 4 Rs. 12 a. a maund. At what price per maund must he sell it to secure a profit of 9 Rs. 6.a. on the whole?

3. How much money must you lay out in the purchase of rice, when it is selling at 4 Rs. 14 a. a maund, in order to be able to realize a profit of 25 Rs. by selling it at 4 Rs. 15 a. a maund?

4. How often will a clock, which strikes the hours, strike in 1877?

5. In a certain manufactory, there are employed a certain number of men, each getting 5 annas a day, and twice as many women, each receiving 3 annas daily. Supposing the total amount of their wages per week to be 57 Rs. 12 a., find the number of men employed.

6. A merchant bought 100 pieces of cloth at 4 Rs. 12 a. per piece. Some pieces were damaged, and he sold the remaining pieces at 5 Rs. per piece, so as just to cover the amount of his outlay. How many pieces were damaged?

VI.

1. What is Compound Division? How do you proceed when the dividend and the divisor are both concrete numbers? How many copies of a book at £1. 15*s.* a copy can be had for £78. 15*s.*?

2. A certain number of men, twice as many women, and three times as many boys, together earned 37 Rs. 3a. a week, each man getting 5 a. each woman 3 a. and each boy 2 a. a day. How many boys were there?

3. A pedestrian who walks 12 kroses a day, started from a certain place some time after another pedestrian, who walks 10 kroses a day, had left the same place. How long after the first did the second pedestrian start, supposing them to meet at the distance of 180 kroses from the place of starting?

4. Light travels at the rate of 192000 miles a second. How long does it take to come from the Sun to the Earth, a distance of 96 millions of miles?

5. Divide 1800 Rs. among A, B and C, so that as often as A gets 2 Rs. B may get 3 Rs. and C 4 Rs.

6. A contractor employs 1 foreman, 2 workmen, and 3 women to do a certain work. They work from the 1st of January to the 30th of April 1876. What is the amount due to them, if the daily wages of a foreman, a workman, and a woman be 8 annas, 6 annas, and 2 annas, respectively? And what would be the contractor's profit, supposing him to get 1 pice in the rupee?

CHAPTER IV.

THE FUNDAMENTAL OPERATIONS WITH CONCRETE FRACTIONS.

SECTION I. NOTATION AND REDUCTION OF CONCRETE FRACTIONS.

185. The method of notation for concrete fractions is the same as that for concrete integers. Thus, *two-thirds of a rupee, three-fifths of an anna, four-tenths of a pice*, will be written $\frac{2}{3}$R., $\frac{3}{5}$a., ·4 pice respectively.

186. The Reduction of concrete fractions is effected by combining the Rules in Art. 173, with the Rules for the Multiplication and Division of fractions and decimals, as will be seen from the following Examples.

Ex. 1. Reduce $\frac{5}{6}$ of a rupee to pies.

$$\begin{aligned}\tfrac{5}{6}\text{R.} &= \tfrac{5}{6} \times 16\text{a.} = 13\tfrac{1}{3}\text{a.}\\ &= 13\text{ a.} + \tfrac{1}{3} \times 12\text{ p.}\\ &= 13\text{ a. } 4\text{ p.} = 160\text{p.}\end{aligned}$$

This Example might also have been worded thus :—

Find the value in pies of $\frac{5}{6}$ of a rupee.

Ex. 2. Find the value of ·005 of £ 1.

$$\begin{aligned}\cdot005 \text{ of } £1 &= \cdot005 \times 20s = \cdot1s.\\ &= \cdot1 \times 12d. = 1\cdot2d.\end{aligned}$$

Ex. 3. Reduce $\frac{2}{3}$ of a pice to the denomination of a rupee.

$$\begin{aligned}\tfrac{2}{3}\text{ pice } &= (\tfrac{2}{3} \div 4)\text{ a.} = \tfrac{1}{6}\text{ a.}\\ &= (\tfrac{1}{6} \div 16)\text{R.} = \tfrac{1}{96}\text{R.}\end{aligned}$$

i.e., $\frac{2}{3}$ of 1 pice $= \frac{1}{96}$ of 1R.

Ex. 4. Reduce 2·5*s.* to the decimal of £1.

$$2\cdot5s. = £(2\cdot5 \div 20) = £\cdot125.$$

Ex. 5. Find the value of $\frac{2}{3}$ of £5.

$$\begin{aligned}\tfrac{2}{3} \text{ of } £5 &= 5 \text{ times } \tfrac{2}{3} \text{ of } £1 = 5 \times £\tfrac{2}{3} = £\tfrac{10}{3} = £3\tfrac{1}{3}\\ &= £3 + \tfrac{20}{3}s. = £3.\ 6s. + \tfrac{2}{3}s.\\ &= £3.\ 6s.\ 4d.\end{aligned}$$

187. RULE I. To reduce one concrete number to the fraction of another concrete number of the same kind, reduce both to the same denomination, and put the former as the numerator and the latter as the denominator.

Ex. Reduce 11 a. to the fraction of 5 Rs. 8a.

By the Rule we have

5 Rs. 8a. $=$ 88a.,

$\therefore$ the fraction reqd. $= \frac{11}{88} = \frac{1}{8}$.

Reason for the Rule.

The question is in other words to find a fraction such, that if 5 Rs. 8a., regarded as the unit, be divided into a number of parts as denoted by its denominator, and a number of these parts be taken as indicated by its numerator, the result will be equal to 11a. Now, our unit 5 Rs. 8 a. *i. e.*, 88a. being divided into 88 parts, each part is 1 anna, and 11 such parts will be 11 a.; $\therefore$ $\frac{11}{88}$ *i. e.*, $\frac{1}{8}$ is the fraction required.

The same Example may be also worded thus :—

What part or fraction of 5 Rs. 8a. is 11a.?

The answer will be $\frac{11}{88}$, or $\frac{1}{8}$ for, evidently 11a. is $\frac{11}{88}$ths of 5 Rs. 8a. or 88 a. regarded as the unit.

RULE II. To reduce one concrete number to the decimal or another concrete number of the same kind, reduce the former to the fraction of the latter and then reduce the fraction so obtained to its corresponding decimal.

Ex. Reduce 6*d.* to the decimal of £5.

By the Rule we have

$$£5 = 240 \times 5d. = 1200d.$$

$$\therefore \text{ the fraction} = \tfrac{6}{1200} = \tfrac{1}{200}$$

and $\therefore$ the decimal $= \cdot 005$.

188. We can compare the values of concrete fractions by first reducing them to a common denomination and then comparing them as abstract fractions.

Ex. Compare the values of $\frac{5}{18}$ *of* 1 R. and $\frac{3}{5}$ of 8a.

We have $\frac{5}{18}$ R. $= \frac{5}{18} \times 16$a. $= \frac{40}{9}$a.

and $\frac{3}{5}$ of 8a. $= \frac{3}{5} \times 8$a. $= \frac{24}{5}$a. $= \frac{4}{5}$a.

Hence $\frac{3}{5}$ of 8a. is greater than $\frac{5}{18}$ of 1 R.

Ex. XXIX.

1. Find the values of

(1) $\frac{3}{5}$ of £ 1. 17*s.* 6*d.*; and ·5 of £ 6. 7*s.* 4*d.*
(2) $\frac{6}{7}$ of £15. 1*s.* 7*d.*; and ·25 of £ 10. 0*s.* 8*d.*
(3) $\frac{3}{8}$ of £ 10. 10*s.* 8*d.*; and ·6 of £ 56. 5*s.*
(4) $\frac{5}{6}$ of £ 9. 9*s.* 6*d.*; and ·4 of £66. 10*s.*
(5) $\frac{2}{3}$ of 4 Rs. 5a. 4p.; and ·8 of 6 Rs. 4a. 10p.
(6) $\frac{7}{10}$ of 8 Rs. 2a. 10p.; and ·2 of 7 Rs. 3a. 5p.
(7) $\frac{7}{8}$ of 15 cwt. 2 qrs. 9 lbs.; and ·3 of 5 lbs. 5 oz. 10 dwts.
(8) $\frac{4}{5}$ of 9 mds. 6 seers. 9 chts.; and ·9 of 8 mds. 12 seers. 8 chts.
(9) $\frac{4}{9}$ of 1 mile 6 fur. 7 po.; and ·12 of 15 yds. 2 ft. 11 in.
(10) $\frac{1}{15}$ of 31 days 11 hrs. 15′; and ·75 of 3 hrs. 32′.

2. Reduce

(1) 15*s.* to the fraction of £ 1; and 16*s.* to the decimal of £1.
(2) 9*s.* 6*d.* to the fraction of £ 19. 19*s.*; and 7*s.* 6*d.* to the decimal of £ 1. 10*s.*
(3) 10a. 8p. to the fraction of 1 R.; and 9a. 6p. to the decimal of 1 R. 3a.
(4) 5a. 6p. to the fraction of 6 Rs. 3a.; and 10a. to the decimal of 6 Rs. 4a.
(5) 15 hrs. 30 min. to the fraction of 31 days; and 5 hrs. 15 min. to the decimal of 1 week.
(6) 3 pts. to the fraction of 1 gal.; and 2 pks. to the decimal of 4 bus.
(7) 3 ft. 9 in. to the fraction of 4 yds.; and 9 in. to the decimal of 1 yd.
(8) 8 kathas to the fraction of a bigha; and 4 acres to the decimal of 1 sq. mile.

3. What fraction of £1 is 12*s.* 6*d.*; and what decimal of 1R. is 5a. 4p.?

4. What fraction of a month of 30 days is 18 hrs.; and what decimal of a week is 21 hrs.?

5. What part of 30 Rs. is 7 Rs. 8 a.; and what part of 100 Rs. is 6 Rs. 4 a.?

6. Compare the values of

(1) $\frac{1}{12}$ of 1 R., $\frac{3}{8}$ of 2 a., and $\frac{1}{15}$ of 1 R. 8 a.
(2) $\frac{1}{3}$ of £ 1., $\frac{3}{8}$ of 2 crowns, and $\frac{5}{6}$ of 1 florin.
(3) $\frac{1}{15}$ of 1 md., $\frac{3}{8}$ of 15 seers, and $\frac{7}{8}$ of 1 pusury.
(5) $\frac{1}{15}$ of 2 yds., $\frac{5}{8}$ of 4 ft., and $\frac{5}{9}$ of 3 ft.

SECTION II. ADDITION OF CONCRETE FRACTIONS.

189. The Addition of concrete fractions may be performed by either of two ways, *viz.*,

First, By finding the values of the several fractions and then adding these values; or

Secondly, By reducing the fractions to a common denomination, then adding them as abstract fractions, and lastly finding the value of the sum.

Sometimes the first method will be found convenient, and sometimes the second, as will be seen from the Examples given below.

Ex. 1. Add together $\frac{2}{5}$ of £ 10. 10*s.*, $\frac{1}{3}$ of 6*s.* 9*d.*, and $\frac{5}{7}$ of £ 14. 1*s.* 2*d.*

Here we follow the first method.

$\frac{2}{5}$ of £ 10. 10*s.* = £ 4. 4*s.*
$\frac{1}{3}$ of 6*s.* 9*d.* = 2*s.* 3*d.*
$\frac{5}{7}$ of £14. 1*s.* 2*d.*
= £ 10 + $\frac{5}{7}$ of 1*s.* 2*d.*
= £ 10 + $\frac{5}{7}$ of 14*d.* = £ 10. 0*s.* 10*d.*

∴ the sum reqd. = £ 14. 7*s.* 1*d.*

Ex. 2. Find the sum of $\frac{2}{27}$ of 3 Rs., $\frac{1}{9}$ of 8 a., and $\frac{2}{21}$ of 1 R. 12 a.

Here we follow the second method.

$\frac{2}{27}$ of 3 Rs. $= \frac{2}{27} \times 3 \times 16 = \frac{32}{9}$ a.
$\frac{1}{9}$ of 8 a. $= \frac{8}{9}$ a.
$\frac{2}{21}$ of 1 R. 12 a. $= \frac{2}{21} \times 28$ a. $= \frac{8}{3}$ a.

$$\therefore \text{ the sum} = \left(\frac{32}{9} + \frac{8}{9} + \frac{8}{3}\right) = \frac{64}{9} \text{ a.} = 7\tfrac{1}{9} \text{ a.} = 7\text{a. } 1\tfrac{1}{3}\text{p.}$$

Ex. 3. Find the sum of

$\dot{3}$ of £ 2., ·5 of 10s. and ·025 of £8.

$\cdot\dot{3}$ of £ 2 $= \frac{1}{3}$ of £ 2 $=$ £ $\frac{2}{3}$.
·5 of 10s. $= \frac{5}{10}$ of £ $\frac{1}{2} =$ £ $\frac{1}{4}$.
·025 of £ 8 $=$ £$(\frac{25}{1000} \times 8) =$ £ $\frac{1}{5}$.

$$\therefore \text{ the sum reqd.} = £\left(\frac{2}{3}+\frac{1}{4}+\frac{1}{5}\right) = £\ \frac{40+15+12}{60}$$
$$= £\,1 + \tfrac{7}{60} \times 20s.$$
$$= £\,1.\ 2s.\ 4d.$$

Ex. XXX.

Find the value of

(1) £ $\frac{1}{2} + \frac{1}{3}$ s. $+ \frac{3}{7}$ guin. $+ \frac{3}{5}$ crown.
(2) £ $\frac{3}{8} + \frac{7}{10}$ s. $+ \frac{7}{8}$ of $\frac{3}{7}$ of £ 1 $+ 3\frac{1}{3}s$.
(3) $\frac{2}{3}$ R. $+ \frac{3}{15}$ of 5 a. $+ \frac{3}{5}$ of 7 Rs.
(4) ·5 s. + £ $\cdot\dot{3}$ + 3 $\cdot\dot{3}s$. + £ $\frac{2}{5}$.
(5) £ $\cdot\dot{1}$ + ·2s. + ·3d.
(6) ·3 R. + ·4 a. + ·5 of 6 Rs. 10 a.
(7) $\frac{1}{3}$ of 1 md. 2 seers. 15 chts. $+ \frac{1}{12}$ of 3 mds. $+ \frac{7}{16}$ of 4 mds.
(8) $\frac{2}{3}$ of 1 ft. 6 in. $+ \frac{3}{4}$ of 2 ft. 3 in. $+ \frac{1}{5}$ of 1 ft. 3 in.
(9) $\frac{1}{7}$ of 2 wks. 21 hrs. $+ \frac{1}{3}$ of 2 days $+ \frac{1}{6}$ of 7 days.
(10) $\frac{1}{8}$ of 2 cwt. $+ \frac{3}{8}$ of 15 lbs. $+ \frac{3}{11}$ of 22 lbs.

SECTION III. SUBTRACTION OF CONCRETE FRACTIONS.

190. The Subtraction of concrete fractions may be performed in either of the two ways mentioned in Art. 189.

Ex. 1. Subtract $\frac{2}{5}$ of 10 Rs. 10 a. from $\frac{3}{11}$ of 22 Rs.

$\frac{3}{11}$ of 22 Rs. $=$ 6 Rs.
$\frac{2}{5}$ of 10 Rs. 10 a. $=$ 4 Rs. 4 a.

$\therefore$ the difference $=$ 1 R. 12 a.

Ex. 2. Subtract $\frac{2}{7}$ of 2 Rs. from $\frac{3}{5}$ of 3 Rs.

$$\begin{aligned} \text{Difference reqd.} &= (\tfrac{3}{5} \times 3 - \tfrac{2}{7} \times 2) \text{ Rs.} \\ &= (\tfrac{9}{5} - \tfrac{4}{7}) \text{ Rs.} = \tfrac{43}{35} \text{ Rs.} \\ &= 1 \text{ R.} + \tfrac{8}{35} \times 16 \text{ a.} \\ &= 1 \text{ R. } 3 \text{ a.} + \tfrac{23}{35} \times 4 \text{ pice.} \\ &= 1 \text{ R. } 3 \text{ a. } 2\tfrac{22}{35} \text{ pice.} \end{aligned}$$

Ex. 3. Subtract ·05 of £ 1 from ·5 of 4*s.*

$$\begin{aligned} \cdot 05 \text{ of £ } 1 &= (\cdot 05 \times 20)s. = 1s. \\ \cdot 5 \text{ of } 4s. &= (\tfrac{5}{10} \times 4)s. \quad = 2s.; \\ \therefore \text{ difference } &= (2 - 1)s. \quad = 1s. \end{aligned}$$

Ex. XXXI.

Find the difference between

(1) $\frac{1}{2}$ of £ 3 and $\frac{3}{4}$ of £ 2.
(2) ·5 of £5. 10*s.* and ·25 of £10. 5*s.*
(3) $\frac{7}{9}$ of £3. 12*s.* and $\frac{3}{7}$ of £15. 1*s.*
(4) $\frac{2}{3}$ of 1 R. 4a. and $\frac{3}{5}$ of 6 Rs. 9a.
(5) $\frac{1}{8}$ of 5 Rs. 4a. and $\frac{1}{12}$ of 9 Rs. 10a.
(6) ·2 of 5 Rs. and $\cdot\dot{3}$ of 6 Rs.
(7) $\frac{2}{3}$ of 1cwt. 14 lbs. and $\frac{1}{4}$ of 2 qrs. 12 lbs.
(8) ·5 of 2ft. 4in. and $\frac{1}{5}$ of 6ft. 8in.
(9) ·24 of 3mds. 5 seers and $\frac{1}{3}$ of 1md. 2 seers.
(10) $\frac{1}{12}$ of 15hrs. 16′ and ·75 of 5hrs.

SECTION IV. MULTIPLICATION OF CONCRETE FRACTIONS.

191. The multiplier alone, or the multiplicand, or both may involve fractions.

Rule. I. If the multiplier alone involves fractions, reduce it to the form of an improper fraction, if necessary, divide the multiplicand by the denominator of this fraction, and then multiply the quotient by the numerator.

Rule II. If the multiplicand or both the multiplicand and multiplier involve fractions, reduce the multiplicand to one denomination, then multiply the numbers as abstract fractions, and then find the value of the result.

Ex. 1. Multiply £6. 7*s*. 8*d*. by $\frac{2}{3}$.

By the Rule we have

$$
\begin{array}{r|lll}
3) & £6. & 7s. & 8d. \\
\hline
 & 2. & 2. & 6\frac{2}{3} \\
 & & & 2 \\
\hline
 & £4. & 5s. & 1\frac{1}{3}d.
\end{array}
$$

Reason for the Rule.

To multiply a number by a fraction is to divide it by the denominator and then multiply the quotient by the numerator (Art. 101). Therefore to multiply the given concrete number by $\frac{2}{3}$, we divide it by 3, and then multiply the quotient by 2.

Ex. 2. Multiply $\frac{5}{48}$ R. + $\frac{3}{5}$ a. by $3\frac{3}{4}$.

$$\tfrac{5}{48}\text{ R.} = \tfrac{5}{48} \times 16\text{ a.} = \tfrac{5}{3}\text{ a};$$

$$\therefore \text{multiplicand} = \left(\tfrac{5}{3} + \tfrac{3}{5}\right)\text{ a.} = \tfrac{34}{15}\text{ a.},$$

$$\text{and multiplier} = \tfrac{15}{4};$$

$$\therefore \text{product} = \tfrac{34}{15} \times \tfrac{15}{4}\text{ a.}$$

$$= \tfrac{17}{2}\text{ a.}$$

$$= 8\text{ a. } 6\text{ p.}$$

Ex. 3. Multiply $\frac{3}{5}$ of 1R. by ·25.

$$\text{Product} = \left(\tfrac{3}{5} \times \cdot 25\right)\text{ R.} = \tfrac{3}{20}\text{ R.}$$

$$= \cdot 15\text{ R.}$$

Ex. XXXII.

Find the value of

(1) £1. 5*s*. 7*d*. × $\frac{2}{3}$.

(2) £15. 13*s*. 6*d*. × $\frac{4}{9}$

(3) £28. 7*s*. 7*d*. × ·25.

(4) 3Rs. 5a. 6p. × $\frac{3}{8}$.

(5) 57Rs. 14a. 10p. × $\frac{3}{10}$.

(6) 25Rs. 15a. 9p. × ·75.

(7) ($\frac{3}{5}$cwt. + $\frac{2}{7}$qrs.) × $\frac{3}{4}$.

(8) ($\frac{17}{20}$md. + $\frac{3}{5}$ seer) × $\frac{7}{16}$.

(9) (£·5. + ·5*s*.) × ·025.

(10) (·3R. + 15a.) × $\frac{7}{16}$.

(11) ($\frac{2}{3}$yd. + $\frac{1}{6}$ft.) × $\frac{5}{9}$.

(12) (£·02. + ·11*s*.) × $33\frac{1}{3}$.

SECTION V. DIVISION OF CONCRETE FRACTIONS.

192. RULE I. If the dividend is wholly integral and the divisor is an abstract fraction, reduce it to the form of an improper fraction, if necessary, and then divide the dividend by the numerator, and multiply the quotient by the denominator.

RULE II. In other cases, reduce the dividend, and, if necessary, the divisor, to the same denomination; then divide as in the division of abstract fractions, and then find the value of the result, when it is a concrete number.

Ex. 1. Divide £10. 6*s.* 6*d.* by $\frac{3}{4}$.

By Rule I, we have

$$\begin{array}{r} 3)\ £10.\quad 6s.\quad 6d. \\ \hline £3.\quad 8s.\quad 10d. \\ 4 \\ \hline £13.\quad 15s.\quad 4d. \end{array}$$

Reason for the Rule.

To divide a number by a fraction is to divide it by the numerator and then to multiply the quotient by the denominator (Art. 105).

Therefore to divide the given concrete number by $\frac{3}{4}$, we divide it by 3 and then multiply the quotient by 4.

Ex. 2. Divide $\frac{2}{3}$ of £1 by $\frac{5}{6}$.

By Rule II, we have

$$\tfrac{2}{3} \text{ of } £1 \div \tfrac{5}{6} = £\left(\tfrac{2}{3} \div \tfrac{5}{6}\right) = £\tfrac{4}{5}$$
$$= 16s.$$

Ex. 3. Divide £$\frac{2}{5}$ + $\frac{3}{5}$*s.* by $\frac{6}{7}$*d.*

By Rule II, we have

$$\left(£\tfrac{2}{5} + \tfrac{3}{5}s.\right) \div \tfrac{6}{7}d.$$
$$= \left(8 + \frac{3}{5}\right)s. \div \frac{6}{7 \times 12}s. = \left(\frac{43}{5} \div \frac{1}{14}\right)$$
$$= \frac{602}{5} = 120\frac{2}{5}.$$

Ex. XXXIII.

Find the value of

1. £10. 19s. 11d. $\div \frac{4}{5}$.
2. £27. 10s.* $\div \frac{2}{3}$ of 11s.
3. 25Rs. 4a. $\div 33\frac{2}{3}$.
4. $17\frac{3}{4}$Rs. $\div$ 4Rs. 7a.
5. 9Rs. 15a. $\div \cdot 6$.
6. 15ft. 6in. $\div 3\frac{1}{10}$.
7. 12ft. 8in. $\div 1\cdot 9$.
8. $\frac{2}{3}$ft. $\frac{4}{5}$in. $\div$ 1ft. 2in.
9. 11mds. 10srs. $\div 1\frac{1}{8}$md.
10. $\frac{2}{3}$cwt. 1qr. $\div$ 2cwt. 3qrs.
11. 10bus. $\frac{1}{3}$pk. $\div 5\frac{1}{2}$.
12. $\cdot$5hrs. 1min. $\div$ 15′. 30″.

SECTION VI. CONVERSION OF CONCRETE NUMBERS.

193. We shall here give some Examples of the conversion of concrete numbers expressed in units of one Table to their equivalents in units of another.

The Rules given in this and the preceding Chapters will be sufficient for the purpose.

Ex. 1. Convert £252. 9s. 9d. to Indian money, supposing 1R. = 2s.

$$\begin{aligned} \text{We have } £1 &= 10\text{Rs.} \\ \therefore £252 &= 2520\text{Rs.} \\ 9s. &= 4\text{Rs. } 8\text{a.} \\ 9d. &= \tfrac{3}{4}s = \tfrac{3}{8}\text{R.} = 6\text{a.} \\ \therefore £252.\ 9s.\ 9d. &= 2524\text{Rs. } 14\text{a.} \end{aligned}$$

Ex. 2. Convert 194Rs. 4a. to English money, supposing 1R. = 2s.

$$\begin{aligned} 194\text{Rs.} &= 388s. = £19.\ 8s. \\ 4\text{a.} &= \tfrac{1}{4}\text{R.} = \tfrac{1}{2}s. = 6d. \\ \therefore 194\text{Rs. } 4\text{a.} &= £19.\ 8s.\ 6d. \end{aligned}$$

Ex. 3. Convert 325 Sicca Rs. to Current rupees.

Since 15 S. Rs. = 16Rs.,

$\therefore$ 1 S. R. = $\frac{16}{15}$Rs.;

$$\text{and} \therefore 325 \text{ S. Rs.} = 325 \times \tfrac{16}{15}\text{Rs.}$$
$$= \frac{65 \times 16}{3}\text{Rs.}$$
$$= \frac{1040}{3}\text{Rs.}$$
$$= 346 \text{ Rs. } 10\text{a. } 8\text{p.}$$

Ex. 4. Convert 325Rs. to Sicca rupees.

$$1\text{R.} = \tfrac{15}{16} \text{ S. R.}$$
$$\therefore 325\text{Rs.} = \tfrac{15}{16} \times 325 \text{ S. Rs.}$$
$$= \frac{4875}{16} \text{ S. Rs.}$$
$$= 304\tfrac{11}{16} \text{ S. Rs.}$$

Ex. 5. Convert 8cwt. 2qrs. 16 lbs. to Indian Bazar weight, and also to pounds Troy.

$$8\text{cwt. } 2\text{qrs. } 16 \text{ lbs.} = 968 \text{ lbs.}$$
$$\text{Now, } 1 \text{ seer} = \tfrac{72}{35} \text{ lbs. Avoir.}$$
$$\therefore 1 \text{ lb. Avoir.} = \tfrac{35}{72}\text{seer,}$$
$$\text{and} \therefore 968 \text{ lbs.} = 968 \times \tfrac{35}{72}\text{seers.}$$
$$= \frac{121 \times 35}{9}\text{seers.}$$
$$= 11\text{mds. } 30\tfrac{5}{9}\text{srs.}$$
$$\text{Again, } 1 \text{ lb. Avoir.} = \tfrac{175}{144} \text{ lbs. Troy.}$$
$$\therefore 968 \text{ lbs. Avoir.} = 968 \times \tfrac{175}{144} \text{ lbs. Troy.}$$
$$= \frac{21175}{18} \text{ lbs. Troy.}$$
$$= 1176\tfrac{7}{18}\text{lbs. Troy.}$$

Ex. 6. Convert 12mds. 15 seers to Factory weight, and also to Troy weight.

$$12\text{mds. } 15\text{srs.} = 12\tfrac{3}{8}\text{mds.}$$
$$= \frac{99}{8}\text{mds.}$$
$$\text{Now, } 1\text{md.} = \frac{54}{49}\text{Factory md.}$$
$$\tfrac{99}{8}\text{mds.} = \tfrac{54}{49} \times \tfrac{99}{8}\text{Fact. mds.}$$
$$= \frac{27 \times 99}{196}\text{Fact. mds.}$$

$$= 13\tfrac{125}{196} \text{ Factory maunds.}$$

Again, $\quad 1 \text{md.} = 100\text{lbs. Troy.}$

$\therefore \quad \frac{99}{8} \text{mds.} = \frac{99}{8} \times 100\text{lbs. Troy.}$

$$= \frac{99 \times 25}{2} \text{lbs. Troy.}$$

$$= 1237\tfrac{1}{2}\text{lbs. Troy.}$$

Ex. 7. Convert 2 sq. miles and 2 miles square to Bengal bighas.

$2 \text{ sq. miles} = 2 \times 640 \text{ acres,}$

and $\quad 1 \text{ acre} = 3\frac{1}{40} \text{ bghs.}$

$= \frac{121}{40} \text{ bghs;}$

$\therefore \quad 2 \text{ sq. miles} = 2 \times 640 \times \frac{121}{40} \text{ bghs.}$

$= 32 \times 121 \text{ bghs.}$

$= 3872 \text{ bghs.}$

Again, $2 \text{ miles square} = 2 \times 2 \text{ sq. miles}$

$= 2 \times 2 \times 640 \text{ acres}$

$= 2 \times 3872 \text{ bghs.}$

$= 7744 \text{ bghs.}$

Ex. 8. Convert 27 bghs. 15 kathas of Bengal measure to Benares bighas.

$27 \text{ bghs. } 15 \text{ kths.} = 27\frac{3}{4} \text{ bghs.} = \frac{111}{4} \text{ bghs.}$

$= \frac{111}{4} \times 1600 \text{ sq. yds.}$

Now, $1 \text{ Benares bigha} = 3136 \text{ sq. yds.}$

$\therefore \quad 27 \text{ bghs. } 15 \text{ kths.} = \frac{111}{4} \times 1600 \div 3136 \text{ Benares bghs.}$

$= \frac{111}{4} \times \frac{1600}{3136} \text{ Benares bghs.}$

$= \frac{111}{4} \times \frac{100}{196} \text{ Benares bghs.}$

$= \frac{111}{4} \times \frac{25}{49} \text{ Benares bghs.}$

$= \frac{2775}{196} \text{ Benares bghs.}$

$= 14\frac{31}{196} \text{ Benares bghs.}$

Ex 9. How many sq. bighas are there in 1 sq. kros?

Since $1 \text{ kros} = 8000 \text{ hands,}$

$= 100 \text{ linear bighas.}$

$\therefore \quad 1 \text{ sq. kros} = 100 \times 100 \text{ sq. bighas.}$

$= 10000 \text{ sq. bighas.}$

Ex. XXXIV.

Convert

1. 525 Sicca Rs. to current rupees.
2. 630 Sicca Rs. to current rupees.
3. 1000 Sicca Rs. to current rupees.
4. 100 Rs. 12a. to Sicca rupees.
5. 250 Rs. 10a. to English money, supposing 1 R. = 2*s*.
6. £ 15. 10s. to Indian money, supposing £1 = 10Rs. 8a.
7. 7 mds. 35 seers to Factory weight, and also to Avoirdupois weight.
8. 1 md. 2 seers to pounds Troy, and also to pounds Avoirdupois
9. 14 lbs. 7 oz. Troy to lbs. Avoirdupois.
10. 16 lbs. 8 oz. Avoir. to lbs. Troy.
11. 21 sq. miles to bighas.
12. 440 bighas 10 kathas to acres.
13. 12 acres 3 roods to bighas.
14. 15 hrs. 15 min. to dandas, and 15 dandas 15 pals to hours.
15. 2 dandas 8 pals to hours, and 9 hrs. 10 min. to dandas.

Miscellaneous Questions and Examples.

194. We shall now give some Examples depending on the preceding Chapters.

Ex. 1. Find the value of

$$\frac{1+\frac{2}{3}+\cdot 2}{\cdot\dot{7}} \text{ of } 6 \text{ Rs.} - \frac{7}{8} \text{ of } 9\text{a.} + \frac{10}{11} \text{ of } 12 \text{ Rs. } 6\text{a.}$$

The given quantity

$$= \left\{ \frac{\frac{5}{3}+\frac{2}{10}}{\frac{7}{9}} \times 6 \times 16 - \frac{63}{8} + \frac{10}{11} \times (12 \times 16 + 6) \right\} \text{a.}$$

$$= \left(\frac{12}{5} \times 6 \times 16 - \frac{63}{8} + 180 \right) \text{a.}$$

$$= \left(\frac{1152}{5} - \frac{63}{8} + 180 \right) \text{a.}$$

$$= \left(230\frac{2}{5} - 7\frac{7}{8} + 180 \right) \text{a.}$$

$$= 402\tfrac{21}{40} \text{ a.}$$

$$= 25 \text{ Rs. } 2\text{a. } 6\cdot 3\text{p.}$$

Ex. 2. A man gives $\frac{1}{3}$ of what he has with him to A; $\frac{1}{4}$ of what remains to B; and 6 annas to C, and finds that he has 1R. 2a. left. How much had he at first, and how much did A and B each get?

A gets $\frac{1}{3}$ of the whole; $\therefore$ there is left $1-\frac{1}{3}$ or $\frac{2}{3}$ of the whole;
and $\therefore$ B gets $\frac{1}{4}$ of $\frac{2}{3}$ or $\frac{1}{6}$ of the whole,
and then what is left $=1-(\frac{1}{3}+\frac{1}{6})=\frac{1}{2}$ of the whole;
and this must be equal to what is given to C together with what is left after all;
i. e., 6a. + 1R. 2a. $=\frac{1}{2}$ of the whole sum;
$\therefore$ the whole sum $= 2\times$ (6a. + 1R. 2a.)
$=$ 3 Rs.
Consequently A gets $\frac{1}{3}$ of 3Rs. or 1 R.,
and B gets $\frac{1}{4}$ of 2 Rs. or 8a.

Ex. 3. Divide £145. 4*s*. among A, B, and C, in such a manner that as often as A gets £3, B shall get £4, and C, £5.

By the question,

for every sum of £ $(3+4+5)$ or £12, A gets £3, B gets £4, and C gets £5; $\therefore$ for every £1, A gets $\frac{3}{12}$ of £1, B gets $\frac{4}{12}$ of £1, and C gets $\frac{5}{12}$ of £1;

and $\therefore$ of the given sum, A gets $\frac{3}{12}$, B, $\frac{4}{12}$, and C, $\frac{5}{12}$; *i. e.*, A gets $\frac{3}{12}$ of £145. 4*s*. or £36. 6*s*., B gets $\frac{4}{12}$ of £145. 4*s*. or £48. 8*s*., and C gets $\frac{5}{12}$ of £145. 4*s*., or £60. 10*s*.

Ex. 4. Divide 590 Rs. among A, B, and C, in such a manner that as often as A gets 3 Rs., B shall get 4 Rs., and as often as B gets 5 Rs., C shall get 6 Rs.

For every 3 Rs. that A gets, B gets 4 Rs.
$\therefore$ .... 1 R. ...... A ...... B...... $\frac{4}{3}$ Rs.

$\therefore$ B's share $=\frac{4}{3}$ of A's share.
Similarly C's...... $=\frac{6}{5}$ of B's share
$=\frac{6}{5}$ of $\frac{4}{3}$ of A's share
$=\frac{8}{5}$ of A's share.

$\therefore$ A's share + B's share + C's share
$=A$'s share $+\frac{4}{3}$ of A's share $+\frac{8}{5}$ of A's share
$=(1+\frac{4}{3}+\frac{8}{5})$of A's share $=\frac{59}{15}$ of A's share.

But A's share + B's share + C's share

$=$ the whole sum to be divided,

i.e., 590 Rs.

$\therefore \frac{59}{15}$ of A's share $= 590$ Rs.,

and $\therefore$ A's share $= 590$ Rs. $\div \frac{59}{15} = 590$ Rs. $\times \frac{15}{59} = 150$ Rs.

Consequently B's share

$= \frac{4}{3}$ of A's share $= \frac{4}{3}$ of 150 Rs. $= 200$ Rs.;

and C's share $= \frac{8}{5}$ of A's share $= \frac{8}{5}$ of 150 Rs. $= 240$ Rs.

Ex. 5. A can do a piece of work in 2 days, B can do it in 3 days, and C in 4 days, working alone. In what time will they finish it working together?

In 2 days A does the whole work,

$\therefore$ in 1 day A does $\frac{1}{2}$ of the work.

Similarly in 1 day B does $\frac{1}{3}$..............

and in 1 day C does $\frac{1}{4}$..............

$\therefore$ in 1 day A, B, C together will do $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ of the work

i. e., $\frac{13}{12}$..............

$\therefore$ in 12 days..............................13 times the work,

and in $\frac{12}{13}$ of a day..............................the whole work

Ex. 6. A can do half a piece of work in 3 hours, being twice as much as B can do; and A, B and C can together do the whole work in $2\frac{1}{2}$ hours. Shew that C can do in 5 hours as much work as B can do in 9 hours.

Here, in 1 hr. A can do $\frac{1}{3}$ of $\frac{1}{2}$ of the work

or $\frac{1}{6}$..................

and............B........$\frac{1}{2}$ of $\frac{1}{6}$ or $\frac{1}{12}$.........

$\therefore$ in 1 hr. A and B together can do $\frac{1}{6} + \frac{1}{12}$ or $\frac{1}{4}$ of the work,

and in 1 hr. A, B and C together can do

$$\frac{1}{2\frac{1}{2}} \text{ or } \frac{2}{5} \text{ of the work}$$

$\therefore$................C can do $\frac{2}{5} - \frac{1}{4}$ ....

i. e., $\frac{3}{20}$ ....

$\therefore$ in 5 hrs. C ....$5 \times \frac{3}{20}$ or $\frac{3}{4}$.........

and in 9 hrs. B ....$9 \times \frac{1}{12}$ or $\frac{3}{4}$.........

$\therefore$ C can do in 5 hrs. as much work as B can do in 9 hrs.

Ex. 7. If an area of 1200 sq. ft. is to be plastered with a mixture of 3 parts of Portland cement, 3 parts of sand, and 1 part of lime, supposing 1 sq. ft. to require $1\frac{1}{3}$ seers of the mixture, how much of each ingredient will be required?

The total quantity of plastering mixture required $= 1200 \times 1\frac{1}{3}$ seers $= 1200 \times \frac{4}{3} \times \frac{1}{40}$ mds. $=$ 40mds.

Now, in the mixture, the whole no. of parts $= 3 + 3 + 1 = 7$ and of these 3 are Portland cement, 3 sand, and 1 lime;

$\therefore$ the quantity of Portland cement

$= \frac{3}{7}$ of 40 mds. $=$ 17 mds. $5\frac{5}{7}$ seers;

the quantity of sand $= \frac{3}{7}$ of 40 mds.

$=$ 17 mds. $5\frac{5}{7}$ seers.

and..............lime $= \frac{1}{7}$ of 40 mds.

$=$ 5 mds. $28\frac{4}{7}$ seers.

Ex. 8. An insolvent debtor owes three creditors *A*, *B* and *C*, 4000 Rs., 5000 Rs., and 7000 Rs. respectively, and his assets amount to 2000 Rs.; how much does each creditor get and what do they get in the rupee?

The debts amount to $(4000 + 5000 + 7000)$ Rs. or 16000 Rs.

$\therefore$ each creditor gets $\frac{2000}{16000}$ R. or $\frac{1}{8}$ R. or 2a. in the rupee;

and $\therefore$ *A* gets $\frac{1}{8}$ of 4000 Rs., or 500 Rs.;

B gets $\frac{1}{8}$ of 5000 Rs., or 625 Rs.;

and *C* gets $\frac{1}{8}$ of 7000 Rs., or 875 Rs.

Ex. 9. A corn-dealer has a mixture of 3 mds. of gram at 1 R. 10 a. a maund, 4 mds. at 1 R. 12 a. a maund, and 5 mds. at 1 R. 14 a. a maund. At what price must he sell the mixture (1) in order to secure a profit of 3. Rs.; and (2) in order to secure the cost price after keeping 2 mds. for his own use?

		R.	a.		Rs.	a.
3 mds.	at	1	10	cost	4	14
4 ...	...	1	12	...	7	
5 ...	...	1	14	...	9	6
$\therefore$ 12 mds.				cost	21	4.

(1) To secure a profit of 3 Rs., he must sell the 12 mds. for 21 Rs. 4 a. + 3 Rs. or 24 Rs. 4a.;

$\therefore$ selling price per maund $=$ 24 Rs. 4 a. $\div$ 12

$=$ 2 Rs. 0 a. 4 pies.

(2) After keeping 2 mds. for his own use, he must sell the remaining 10 mds. for 21 Rs. 4 a.;

$\therefore$ selling price per maund $=$ 21 Rs. 4 a. $\div$ 10

$=$ 2 Rs. 2 a.

Ex. 10. A cistern has 3 pipes A, B, and C. A and B can fill it in 3 and 4 hours respectively; and C can empty it in 1 hour. If these pipes be opened in order at 3, 4, and 5 o'clock, when will the cistern be empty?

In 1 hour A can fill $\frac{1}{3}$ of the cistern,
B......... $\frac{1}{4}$...............
and C can empty the whole of the cistern;
$\therefore$ at 5 o'clock,
A being open for 2 hours has filled $\frac{2}{3}$ of the cistern.
B.......................1 hr.................. $\frac{1}{4}$ ....
i. e., at 5 o'clock, $\frac{2}{3}+\frac{1}{4}$ or $\frac{11}{12}$ of the cistern is full.
Now after 5 o'clock A and B together tend to fill $\frac{1}{3}+\frac{1}{4}$ or $\frac{7}{12}$ of the cistern every hour, and C tends to empty the whole cistern per hour;
$\therefore$ A, B and C being open at the same time, the rate of emptying is $1-\frac{7}{12}$ or $\frac{5}{12}$ of the cistern per hour;
$\therefore$ $\frac{11}{12}$ of the cistern being the portion that was full, will be emptied in $(\frac{11}{12}\div\frac{5}{12})$ hrs.
i.e., in $\frac{11}{5}$ hrs, or 2 hrs. 12′.

Hence the cistern will be empty at 5 hrs. + 2 hrs. 12′ or 12′ past 7 o'clock.

Ex. XXXV.

I.

1. How far is the silver coinage a legal tender in England and in India, and what is its standard fineness in each of the two countries?

Find the value of a rupee in shillings, taking into account only the weight of pure silver contained in each.

2. A creditor in receiving payment of a certain amount of money in the silver coinage of India, finds that the weight of pure silver it contains is exactly $2\frac{3}{4}$ maunds. What is the amount received?

3. What are the weights of pure silver in 1000 rupees and in 2000 shillings?

4. Convert 525 Sicca rupees into current rupees, and also into English money, at 1*s*. 10*d*. per rupee.

5. A person inherits $\frac{3}{8}$ of an estate yielding 5250 Rs. a year. What is the annual income of his share, and what is his profit, if the Government revenue payable for the whole estate is 1050 Rs., and the cost of collection, 10 Rs. for every 100 Rs. of income realized?

6. A person owns 5 annas 4 pies share of a zemindari yielding an income of 6525 Rs. a year, and he subsequently purchases a 2 annas 13 gandas, 1 cowry, 1 krant share of the same. What fraction of the estate does he now own, and what is the income of his share?

II.

1. What is the Imperial Standard Yard, and how may it be recovered if lost?

How many sq. yards are there in an acre, and how many in a bigha?

Convert 1 sq. mile into bighas, and 600 bighas into acres.

2. A person starts from a certain place and walks at the rate of 2 miles an hour. After $2\frac{1}{2}$ hours, another person starts from the same place, and walking at a certain rate overtakes the former at the distance of 25 miles from the starting place. At what rate does the second person walk?

3. *A* owns $\frac{5}{8}$ of a zemindari which contains 8225 bighas. He sells $\frac{3}{4}$ of his share to *B*, and ·5 of what remains to *C*. What share of the estate does he still own, and how much land does that share contain?

4. What fraction of a sq. mile is an acre, and what fraction of an acre is a bigha?

5. What decimal of a mile is a chain, and what fraction a mile is a yard?

6. In the Thakbust scale which is 16 inches to the mile, how many feet does an inch represent, and what fraction of an inch will represent a bigha?

III.

1. In a certain district, the road cess is levied at the rate of one-half of an anna for every rupee of rent realized from an

estate, less a deduction at one half of the said rate for every rupee of the revenue payable in respect thereof. What would be the amount of road cess leviable on an estate of which the gross rental is 8975 Rs. and for which the revenue payable is 3520 Rs.?

2. An insolvent debtor whose debts amount to 150000 Rs., has assets just sufficient to pay his creditors 7 a. in the rupee. Find the amount of the assets.

3. Divide £3500 amongst *A*, *B*, and *C*, so that as often as *A* gets £2, *B* shall get £3, and as often as *B* gets £4, *C* shall get £5.

4. Divide 790 Rs. among *A*, *B*, and *C*, in such a manner that *B*'s share shall be $\frac{3}{4}$ of *A*'s, and *C*'s share $\cdot 3$ of *B*'s.

5. What is the Imperial Standard Troy Pound, and how may it be recovered if lost?

What fraction of a maund is a pound Troy, and what fraction of a hundredweight is a maund?

6. A grocer bought sugar of a certain quality at 12 Rs. 4a. a maund, and twice the same quantity at 13 Rs. a maund, and by mixing the two together and selling the mixture at 12 Rs. 14a. per maund made a profit of 12 Rs. 8 a. How much sugar of each kind did he buy?

IV.

1. Find the value of

$$\frac{2 + \frac{\text{[illegible]}}{3} + \cdot 4}{\cdot 7} \text{ of } 12 \text{ Rs.} + \cdot\dot{3} \text{ of } 2\text{a.} - \cdot 125 \text{ of } 1\text{a.}$$

2. *A* can do a piece of work in 4 days working 7 hrs. a day; *B* can do it in 4 days working 6 hrs. a day; and *C* and A can together do it in 2 days working 4 hrs. a day. In how many days will *A*, *B*, and *C* together finish the work by working 1 hr a day?

3. How many revolutions will the wheel of a carriage 4 ft. 3 in. in diameter, make in going over a mile, supposing the circumference of a circle to be $3\cdot1416 \times$ diameter?

4. How much land at 60 Rs. a katha must be given in exchange for 3 bighas 10 kathas of land at 85 Rs. a katha, together with a building on it worth 3500 Rs.?

5. How many yards of cloth at 6a. per yard must be given in exchange for 16 yards of silk at 6 Rs. 4a. a yard?

6. How much of a maund is $\cdot\dot{3} + \cdot 5 + \frac{1}{12}$ of a cwt.?

V.

1. A cistern has three pipes *A*, *B*, and *C*. *A* and *B* can fill it in 4 and 3 hours respectively, and *C* can empty it in 2 hours. When will the cistern be exactly full if the pipes are opened in order at 1, 2, and 3 o'clock respectively?

2. *A*, *B*, and *C* can respectively do a piece of work in 12, 9, and 6 hours respectively. *A* and *B* together work for an hour and then *B* goes away. How long will *A* and *C* take to finish the remaining portion of the work?

3. Give the length of the solar year, and the average length of the Julian year; and find the difference between the two. In how many years would this difference amount to a day?

4. The 1st of January 1872 was a Monday. How many Sundays were there in that year, and how many Saturdays in the month of August of the same year?

5. What fraction of a kros is a mile, and how many Jojans are there between the Moon and the Earth, a distance of 240000 miles?

6. Light travels at the rate of 192000 miles a second, and sound, at the rate of 1142 feet a second. What time would intervene between the seeing of the flash and the hearing of the report of a gun fired at the distance of $2\frac{1}{2}$ miles?

VI.

1. What are the advantages of the Metric System of Weights and Measures?

How many grains are there in a seer of Bazar weight, and in a seer under the Indian Weights and Measures of Capacity Act 1871?

2. What decimal of an English mile is a kilometre?

3. A pedestrian who walks at the rate of $2\frac{1}{2}$ miles an hour, sets out from a place *A* for a place *B*, a distance of 66 miles, at

the same time that another pedestrian who walks at the rate of 88 yards a minute sets out from *B* for *A*. Where will they meet, and how long after starting; and how much earlier ought the former to have started in order to meet the latter midway between *A* and *B*?

4. *A* and *B* can respectively finish a piece of work in 4 days of 9 working hours each and 3 days of 8 working hours each. In how many days will they finish the work if they work together 7 hours 12 minutes a day?

5. A person lays out £45 in spirits at 7*s.* 6*d.* a gallon. He adds water to it, and by selling the mixture at 7*s.* a gallon, finds that he has made a profit of one shilling for every pound of his outlay. How much water did he add?

6. The 1st of January fell on a Monday in 1877. When will it fall on a Monday again?

CHAPTER V.

SQUARE AND CUBIC MEASURE. DUODECIMALS.

SECTION I. SQUARE AND CUBIC MEASURE.

195. Def. The Area of any figure is the quantity of surface contained in it.

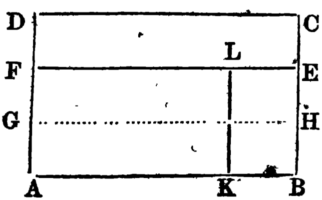
An area is measured by the number of times that it contains another known area which is taken as the *unit of area.* We have seen in Art 148, that the ordinary units of area of several denominations, *viz.*, the square inch, the square foot &c., are the areas of the squares on the ordinary units of length, *viz.*, the linear inch, linear foot, &c.

196. Def. A Rectangle is a four-sided figure having its opposite sides parallel and contiguous sides at right angles to each other.

Any side of a rectangle may be called its length and the contiguous side, its breadth.

Prop. I. If two rectangles have the same length, the area of the one will be the same fraction of that of the other, that the breadth of the former is of the breadth of the latter.

Let ABCD and ABEF be two rectangles having the same length AB, and let AF $= \frac{2}{3}$ AD. Divide AD into 3 equal parts AG, GF, FD; then AF will contain two of these parts. Now drawing the dotted line GH as in the figure, the rectangle ABCD is divided into 3 equal rectangles whereof ABEF contains two; and



$\because$ ABCD $=$ 3 times ABHG,
and ABEF $=$ 2 times ABHG,
$\therefore$ ABEF $=$ $\frac{2}{3}$ ABCD.

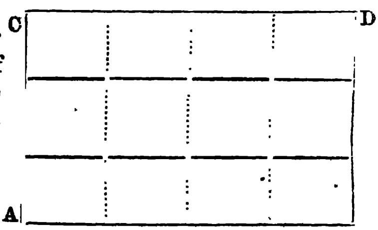
Similarly the Proposition can be proved in other cases.

PROP. II. If the length and the breadth of one rectangle be certain fractions of those of another, the product of these fractions indicates what fraction the area of the former is of that of the latter.

In the above figure, let ABCD and AKLF be two rectangles, and let AK $= \frac{3}{4}$ AB, AF $= \frac{2}{3}$ AD. Then by the preceding Proposition, AKLF $= \frac{3}{4}$ ABEF
$= \frac{3}{4}$ of $\frac{2}{3}$ ABCD.
$= \frac{1}{2}$ ABCD.

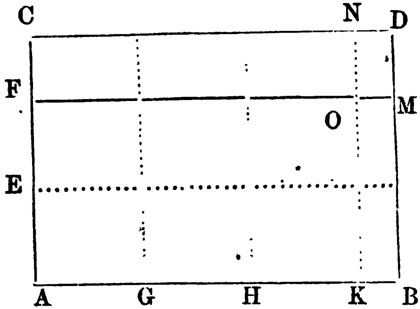
197. PROP. Taking the square on the linear unit as the superficial unit, the number of superficial units in a rectangle = the number of linear units in the length $\times$ the number of linear units in the breadth.

Let ABDC be a rectangle.



First let the number of linear units in each of the sides be integral, say 4 in AB and 3 in AC. Then drawing lines as in the figure, the rectangle ABDC is divided into a number of equal horizontal slips equal to the number of linear units in AC; and each horizontal slip into a number of equal squares or units of superficial measure equal to the number of linear units in AB; and the number of superficial units in ABDC

= the number of horizontal slips
 $\times$ the number of squares in each slip
= the number of linear units in the breadth
 $\times$ the number of linear units in the length
$= 3 \times 4.$



Next let the numbers involve fractions, and let AB contain $3\frac{1}{4}$ and AC $2\frac{2}{3}$ linear units. In AB, AC measure off parts equal to the linear unit; and draw lines through the points of division as in the figure

Then $AE = EF = AG = GH = HK =$ the linear unit,

$FC = \frac{2}{3}$ linear unit $= NO$,

$KB = \frac{1}{4}$ .... $= OM$;

and each of the 3 small rectangles in CFON $= \frac{2}{3}$ of superficial unit, as it has one side equal to a linear unit and the other to $\frac{2}{3}$ of linear unit (Art 196, Prop. I). Similarly each of the 2 small rectangles in BKOM $= \frac{1}{4}$ of superficial unit. And the small rectangle MOND $= \frac{2}{3} \times \frac{1}{4}$ of superficial unit, as it has its side ON $= \frac{2}{3}$ linear unit, and side OM $= \frac{1}{4}$ linear unit (Art. 196, Prop. II).

Hence, the no. of superficial units in AFOK $= 3 \times 2$,

.... CFON $= 3 \times \frac{2}{3}$,

.... BKOM $= 2 \times \frac{1}{4}$,

.... MOND $= \frac{2}{3} \times \frac{1}{4}$;

∴ adding, the no. of superficial units in

AFOK + CFON + BKOM + MOND

i. e., the no. of superf. units in ABDC

$= 3 \times 2 + 3 \times \frac{2}{3} + 2 \times \frac{1}{4} + \frac{2}{3} \times \frac{1}{4}$

$= 3 \times (2 + \frac{2}{3}) + \frac{1}{4} \times (2 + \frac{2}{3})$

$= (3 + \frac{1}{4}) \times (2 + \frac{2}{3}) = 3\frac{1}{4} \times 2\frac{2}{3}$.

Similarly the Proposition can be proved in other cases.

This Proposition is sometimes briefly stated thus :—

Area of a rectangle = length × breadth.

Thus, if the sides of a rectangle are 3 ft. and 4 ft.

its area $= 3 \times 4$ sq. ft.

198. In the preceding Article, it may appear that we have multiplied one concrete number 3 ft. by another 4 ft. and got 12 sq. ft. for the product. But in reality that is not so.

What we have multiplied together are the abstract numbers 3 and 4; and it so happens only from the nature of things, that this product 3×4 represents the number of square feet in the rectangle whose length and breadth are 4 ft. and 3 ft. respectively.

It is only in this sense that we are to understand expressions such as these :—

Feet multiplied by feet give square feet.
Inches................inches.............inches.
&c.........................&c...................&c.

199. A rectangle 5 ft. by 4 inches $= \frac{1}{12}$ of a rectangle 5 ft. by 4 ft.,

$\because$ 4 inches the breadth of one $= \frac{1}{12}$ of 4 ft. the breadth of the other (Art. 196, Prop. I).

Hence 5 ft. $\times$ 4 in. $= \frac{1}{12}$ of 5×4 sq. ft.: and so in other cases.

200. In the Bengal superficial measure,

1 linear bigha $\times$ 1 linear bigha $=$ 1 sq. bigha.
1............bigha $\times$ 1...............katha
$=$ 1............bigha $\times$ $\frac{1}{20}$ of 1...... bigha $= \frac{1}{20}$ of 1 sq. bigha
$=$ 1 sq. katha.
1............katha $\times$ 1...............katha
$= \frac{1}{20}$ of 1....bigha $\times$ 1...............katha $= \frac{1}{20}$ of 1 sq. katha
$=$ 1 *dkool.*

The Bengal method of finding areas will be seen from the following Example :—

Ex. Find the area of a field 5 bighas 4 kathas by 4 bighas 8 kathas.

bghs.	kths.	
5	4	
4	8	
20	16	
2	1	12
22	17	12

The area is 22 sq. bghs 17 sq. kths. 12 dhools.

The reason is obvious.

4 bghs. $\times$ 5 bghs. = 20 sq. bghs.

4 bghs. $\times$ 4 kths. = 16 sq. kths.

5 bghs. $\times$ 8 kths. = 40 sq. kths. = 2 sq. bghs.

8 kths. $\times$ 4 kths. = 32 sq. dhools = 1 sq. kth. 12 sq. dhools.

201. DEF. The VOLUME of a solid is the quantity of space having length, breadth, and thickness, that it contains.

The volume of a solid is measured by the number of times that it contains some other known volume which is taken as the unit of volume. We have seen in Art. 149, that the ordinary units of volume of several denominations, *viz.*, the cubic inch, the cubic foot, &c., are the volumes of the cubes having for their edges the units of length, *viz.*, the linear inch, the linear foot, &c.

202. DEF. A RECTANGULAR PARALLELOPIPED is a solid contained by six rectangles.

Any three contiguous edges of a rectangular parallelopiped may be called its length, breadth, and thickness respectively.

PROP. The volume of a rectangular parallelopiped = length $\times$ breadth $\times$ thickness.

This can be proved in a way similar to that given in Art. 149.

203. Areas and volumes are found by reducing the lengths, breadths, and thicknesses to one and the same denomination, and then performing the necessary multiplications, as will be seen from the subjoined Examples.

Ex. 1. Find the area of a rectangular court-yard 24 ft. 6 in. long and 15 ft. 3 in. broad.

The area = (24 ft. 6 in.) $\times$ (15 ft. 3 in.)

$= (24\frac{1}{2} \times 15\frac{1}{4})$ sq. ft. $= \frac{49}{2} \times \frac{61}{4}$ sq. ft.

$= \frac{2989}{8}$ sq. ft. $= 373\frac{5}{8}$ sq. ft.

= 373 sq. ft. 90 sq. in.

Ex. 2. Find the volume of a cube whose edge is 2 ft. 3 in.

2 ft. 3 in. $= 2\frac{1}{4}$ ft. $= \frac{9}{4}$ ft.;

$\therefore$ the volume $= \frac{9}{4} \times \frac{9}{4} \times \frac{9}{4}$ cub. ft.

$= \frac{729}{64}$ cub. ft.

$= 11\frac{25}{64}$ cub. ft.

= 11 cub. ft. 675 cub. in.

The above Example may also be worked thus :—

2 ft. 3 in. = 27 in.;

$\therefore$ the volume = (27 × 27 × 27) cub. in.
= 19683 cub. in.
= 11 cub. ft. 675 cub. in.

Ex. 3. What length must be cut off from a plank that is ft. 3 in. broad, in order that it may contain 1 sq. yd.?

Since area = length × breadth,

$$\therefore \quad \text{length} = \frac{\text{area}}{\text{breadth}};$$

$$\text{and} \therefore \text{ the reqd. length} = \frac{1 \text{ sq. yd.}}{1 \text{ ft. } 3 \text{ in.}} = \frac{9 \text{ sq. ft.}}{1\frac{1}{4} \text{ ft.}}$$

$$= 9 \times \tfrac{4}{5} \text{ ft.} = 7 \text{ ft. } 2\tfrac{2}{5} \text{ in.}$$

Ex. XXXVI.

1. Find the areas of the following rectangles :—

(1) 2 ft. 3 in. by 1 ft. 6 in. (2) 3 ft. 6 in. by 2 ft. 9 in.
(3) 4 ft. 3 in. by 3 ft. 6 in. (4) 5 ft. 9 in. by 4 ft. 8 in.
(5) 6 ft. 4 in. by 5 ft. 6 in. (6) 7 ft. 8 in. by 1 ft. 4 in.

2. Find the volumes of the following rectangular parallelopipeds :—

(1) 2 ft. 3 in. × 1 ft. 6 in. × 9 in.
(2) 3 ft. 6 in. × 2 ft. 3 in. × 1 ft. 3 in.
(3) 4 ft. 3 in. × 2 ft. 8 in. × 1 ft.
(4) 4 ft. 9 in. × 3 ft. 6 in. × 2 ft.

3. A room is 18 ft. 6 in. by 15 ft. Find the area of its floor, and the length of carpet $1\frac{1}{4}$ yd. wide, that will be required for carpeting the same.

4. A room is 21 ft. 3 in. by 12 ft. 6 in. Find the area of its floor. Find also the length of another room containing the same area, supposing its breadth to be 10 ft.

5. Find the solid content of a brick that is 10 in. long, 5 in. broad, $2\frac{1}{2}$ in. thick, and also that of a wall that is 20 ft. long, 10 ft. 6 in. high, and 2 ft. thick.

6. Find the areas of the following rectangular fields :—

(1) 2 bghs. 8 kths. by 1 bgh. 10 kths.
(2) 3 bghs. 10 kths. by 2 bghs. 5 kths.
(3) 4 bghs. by 3 bghs. 8 kths.
(4) 5 bghs. 6 kths. by 4 bghs. 15 kths.
(5) 10 bghs. 10 kths. by 1 bgh. 14 kths.
(6) 12 bghs. 10 kths. by 3 bghs. 8kths.

SECTION II. DUODECIMALS.

204. Besides the method given in the Art. 203, there is another method of finding areas and volumes, called the method of DUODECIMALS OR CROSS MULTIPLICATION, which is generally employed by painters, brick-layers, &c., in measuring work.

In this method, the successive linear units are, Feet, Primes, Seconds, Thirds, &c., and are so connected that a unit of any denomination is $\frac{1}{12}$ of a unit of the next higher denomination.

Primes, seconds, thirds, &c., are indicated by the accents $'$, $''$, $'''$, &c., placed above the numbers, a little to the right. It is evident that primes are the same as inches.

The successive superficial units in this method are superficial feet, primes, seconds, thirds, &c., and are so connected that a unit of any denomination is $\frac{1}{12}$ of a unit of the next higher denomination. They are indicated in the same way as the linear units.

The successive solid units are solid feet, primes, seconds, thirds, &c. They are connected and indicated in the same way as linear or superficial units.

205. Hence, by Art. 196,

1 ft. $\times 1' =$ 1ft. $\times \frac{1}{12}$ of 1 ft. $= \frac{1}{12}$ of 1 sq.ft. $= 1'$ of square measure
1 ft. $\times 1'' = \frac{1}{144}$ of 1 sq. ft. = 1 sq. in. $= 1''$ ....
1 ft. $\times 1''' = \frac{1}{12}$ of 1 sq. in. $= 1'''$ ....
$1' \times 1' =$ 1 sq. in. $= 1''$ ....
$1' \times 1'' = \frac{1}{12}$ of 1 sq. in. $= 1'''$ ....

Similarly (by Art. 202),

1 sq. ft. $\times 1' = \frac{1}{12}$ of 1 cub. ft. $= 1'$ of solid ....
1 sq. ft. $\times 1'' = \frac{1}{144}$ of 1 cub. ft. $= 1''$ ....
1 sq. ft. $\times 1''' =$ 1 cub. in. $= 1'''$ ....
$1'$ (superf.) $\times 1' = \frac{1}{144}$ of 1 cub. ft. $= 1''$ ....
$1'$ .... $\times 1'' =$ 1 cub. in. $= 1'''$ ....

It will be seen that the denomination of every product is indicated by the sum of the accents of the factors, the number of accents in *feet* being regarded as 0.

206. We shall now give the Rule for finding areas and volumes by the above method.

RULE. Write the terms of the multiplier under the corresponding terms of the multiplicand. Multiply every term of the multiplicand beginning with the lowest, by the highest term of the multiplier, divide each product (except where it is of the denomination of feet) by 12, carry the quotient to be added to the next product, and put down the remainder under the denomination indicated by the sum of the accents in the factors. Proceed in this way with every successive term of the multiplier, placing the partial product corresponding to each term, below the preceding partial product. Add up the several denominations of the partial products, carrying 1 for every 12. The sum will be the product required.

Ex. Multiply 3 ft. 9 in. by 2 ft. 7 in.

By the Rule we have

$$\begin{array}{rrr} 3. & 9' & \\ 2. & 7' & \\ \hline 7. & 6' & \\ 2. & 2. & 3'' \\ \hline 9. & 8' & 3'' \end{array}$$

The *reason for the Rule* will be seen below.

It is evident from Art. 205,

that 2 ft. × 9′ = 18′ (superf.) = 1 sq. ft. + 6′

2 ft. × 3 ft. = 6 sq. ft.

∴ 3 ft. 9 in. × 2 ft. = 7 sq. ft. + 6′.

Again 7′ × 9′ = 63′′ = (60′′ + 3′′) = 5′ + 3′′

7′ × 3 ft. = 21′ = 1 sq. ft. + 9′

∴ 3 ft. 9′ × 7′ = 1 sq. ft. 9′ + 5′. 3′′ = 2 sq. ft. 2′. 3′′.

∴ the reqd. product = 7 sq. ft. 6′ + 2 sq. ft. 2′. 3′′ = 9 sq. ft. 8′. 3′′.

The result may be expressed in sq. feet and inches thus :—

$$\begin{aligned} 9 \text{ sq. ft. } 8'. 3'' &= 9 \text{ sq. ft.} + \left(\tfrac{8}{12} + \tfrac{3}{144}\right) \text{sq. ft.} \\ &= 9 \text{ sq. ft.} + \tfrac{99}{144} \text{ sq. ft.} \\ &= 9 \text{ sq. ft. } 99 \text{ sq. in.} \end{aligned}$$

207. The above method is called the method of Duodecimals, because the number twelve (*duodecim* in Latin) forms the basis of connection between successive denominations; and it is also called the method of Cross Multiplication, because the multiplication is performed in a cross order, *i. e.*, whereas we begin with the highest denomination in the multiplier, we begin with the lowest in the multiplicand.

Ex. XXXVII.

Find by Cross Multiplication the areas of the following rectangles :—

1.	2 ft. 9 in. × 1 ft. 3 in.	2.	3 ft. 4 in. × 4 ft. 8 in.
3.	3 ft. 8 in. × 5 ft. 9 in.	4.	5 ft. 9 in. × 6 ft.
5.	6 ft. 7 in. × 7 ft. 8 in.	6.	6 ft. 8 in. × 7 ft. 7 in.
7.	8 ft. 9 in. × 6 ft. 11 in.	8.	12 ft. 11 in. × 4 ft. 6 in.
9.	9 ft. 9 in. × 3 ft. 5 in.	10.	10 ft. 10 in. × 5 ft. 5 in.

MISCELLANEOUS QUESTIONS AND EXAMPLES.

208. We shall here give some Examples depending upon this and the preceding Chapters.

Ex. 1. What is the cost of building a wall 24 ft. long, 14 ft. high, and 2 ft. 3 in. thick for the first 4 ft. of its height, and 2 ft. thick for the remainder, at 22 Rs. 8a. per 100 cub. ft.? And how many bricks will be required supposing each brick to be 9 in. × 4 in. × 3 in?

The volume of the wall
= 24 ft. × 4 ft. × 2 ft. 3 in. + 24 ft. × 10 ft. × 2 ft.
= $(24 \times 4 \times \frac{9}{4} + 24 \times 10 \times 2)$ cub. ft.
= 696 cub. ft.

Now, 100 cub. ft. cost 22 Rs. 8a. *i.e.*, $\frac{45}{2}$ Rs.;

$\therefore$ 1 cub. ft. will cost $\frac{45}{2} \div 100$ or $\frac{9}{40}$ Rs.,

and $\therefore$ the wall will cost $696 \times \frac{9}{40}$ Rs.
i. e., $\frac{87 \times 9}{5}$ Rs. or 156 Rs. 9a. $7\frac{1}{5}$ p.

Again, the no of bricks reqd.

$$= \frac{\text{volume of the wall}}{\text{volume of a brick}}$$

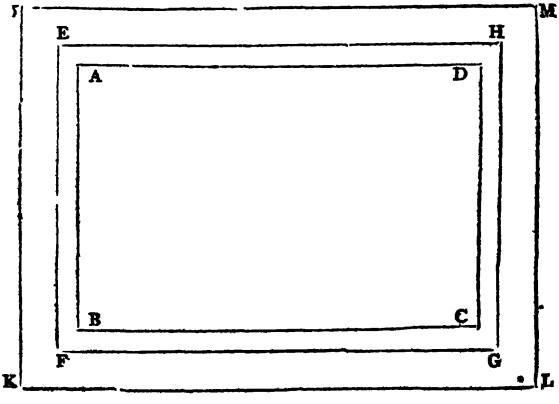
$$= \frac{696}{\frac{1}{2} \times \frac{4}{12} \times \frac{3}{12}} = 696 \times 16$$

$$= 11136$$

Ex 2 A rectangular tank, 1 bigha 10 kathas in length and 1 bigha in breadth, has a rectangular gravel walk 4 cubits broad along its sides, at a distance of 2 cubits from each side. Find the cost of graveling the walk, at the rate of 1R per 100 sq ft.

Let ABCD be the tank, so that AB = 1 bgh., BC = 1 bgh 10 kth., and let the space between EFGH and IKLM be the gravel walk. Then area of the gravel walk

= rectangle IKLM—rectangle EFGH.



Now EF = AB + 2 × dist. of the walk from the tank

= 80 cubits + 2 × 2 cubits

= 84 cubits = 126 ft ;

and FG = BC + 2 × dist. of the walk from the tank,

= (120 + 4) cubits = 186ft.

Again IK = EF + 2 × breadth of the walk

= 126ft. + 12ft. = 138ft ;

and KL $= FG + 2 \times$ breadth of the walk
$= 186\text{ft.} + 12\text{ft.} = 198\text{ft.}$

Therefore area of the walk
$= IK \times KL - EF \times FG$
$= (138 \times 198)$ sq. ft. $- (126 \times 186)$ sq. ft.
$= 3888$ sq. ft.

Now the cost of graveling 100 sq. ft. $= 1$ R.,
$\therefore$ ....1 sq. ft. $= \frac{1}{100}$R.
and $\therefore$ ....3888 sq. ft. $= \frac{1}{100} \times 3888$ Rs.
$= \frac{972}{25}$ Rs.
$= 38\frac{22}{25}$Rs.
$= 38$Rs. 14a. $\frac{24}{25}$p.

Ex. 3. Find the price of a beam 15 ft. long, and 7 in $\times$ 5 in. at its end, at 1 pice per 1 ft. $\times$ 1 in. $\times$ 1 in.

The volume of which the price is 1 pice
$= 1\text{in.} \times 1\text{in.} \times 1\text{ ft.} = 1 \times 1 \times 12$ cub. in. $= 12$ cub. in.

The volume of 1 foot of the beam
$= 7\text{ in.} \times 5\text{ in.} \times 1\text{ ft.} = 7 \times 5 \times 12$ cub. in.
$= 35 \times 12$ cub. in.
$= 35 \times$ the volume of which the price is 1 pice.

Therefore the price of 1 foot of the beam
$= 35$ pice $= 8$ a. 3 pice,

and $\therefore$ the price of the whole beam
$= 15 \times 8$ a. 3 pice.
$= 8$ Rs. 3 a. 1 pice.

Ex. 4. What length must be cut off from a plank that is 9 in. wide in order that it may contain 1 sq. yard?

The area of the plank is required to be 1 sq. yd., and its breadth is 9 in.

Now length $\times$ breadth $=$ area

$$\therefore \text{length} = \frac{\text{area}}{\text{breadth}},$$

nd $\therefore$ the reqd. length

$$= \frac{1 \text{ sq. yd.}}{9 \text{ in.}}$$

$$= \frac{9 \text{ sq. ft.}}{\frac{3}{4} \text{ ft.}}$$

$$= 12 \text{ ft.}$$

Ex. 5. Find the expense of painting the walls of a room 24 ft. in length, 15 ft. in breadth and 12 ft. in height, at 4 a. per sq. yd.

The area of 2 of the walls $= 2 \times$ length $\times$ height
$= 2 \times 24 \times 12$ sq. ft.
............the other 2 walls $= 2 \times$ breadth $\times$ height
$= 2 \times 15 \times 12$ sq. ft.

$\therefore$ the whole area to be painted
$= (576 + 360)$ sq. ft.
$= 936$ sq. ft.
$= 104$ sq. yd.;
and $\therefore$ reqd. expense $= 104 \times 4$ a.
$= 26$ Rs.

Ex. 6. A piece of land measures 6 bgh. 5 kth. in length, and 3 bgh. 18 kth. in breadth: find its area in the Bengali method, and its price at 80 Rs. per katha.

bgh.	kth.	
6	5	
3	18	
18	15	
5	12	10 dhools.
area = 24 bghs.	7 kths.	10 dhools.

Now price of 24 bghs. $= 24 \times 20 \times 80$ Rs. $= 38400$ Rs.
..........7 kths. $= 7 \times 80$ Rs. $= 560$ Rs.
........10 dhools $= \frac{1}{2} \times 80$ Rs. $= 40$ Rs.
$\therefore$ total price $= 39000$ Rs.

Ex. XXXVIII.

1. When one number is multiplied by another, the latter or the multiplier must be an abstract number. Shew that exceptions to this rule are only apparent and not real.

Find the value of 2 mds. 16 seers of sugar at 5 a. 6p. a seer, and explain clearly the steps of your process.

2. What do you mean by saying that the area of a rectangle is equal to the product of the base and the altitude? Illustrate your meaning by an example.

Find the area of a field 5 bghs. 6 kths. long and 4 bghs. 10 kths. broad.

3. What will be the cost of painting the four walls of a room, whose length is 18 ft., breadth 9 ft. 6 in., and height 10 ft., at 1*s.* a sq. yard?

4. A house contains 6 rooms; each room has 3 windows; each window is 7 ft. in height and 3 ft. 6 in. in breadth. Find the expense of painting the windows, at 1 anna a sq. foot.

5. A rectangular piece of land 5 bghs. 10 kths. long, contains an area of 22 sq. bighas. What is its breadth?

6. A piece of land measures 18 kths. by 16 kths., and contains a small tank 90 ft. by 48 ft. What is its value, if the area covered by the tank is worth 60 Rs. a katha, and the remainder, 125 Rs. a katha?

7. Find the expense of paving a rectangular court 24 ft. long and 18 ft. broad, at 1 R. 4 a. per sq. cubit.

8. A gentleman promenading in a veranda observes that in the time between his leaving an end of the veranda and coming back again to it, the minute hand of a clock passes exactly from one minute mark to the next. After pacing to and fro for 20 minutes, he finds that he has walked half a mile. What is the length of the veranda?

9. On buying a rectangular piece of land, the purchaser found that if the price he paid for it, were converted into eight-anna pieces, and the amount spread on the land in rows in contact with one another, it would have just covered the land. What was the price per katha, supposing an eight-anna piece to be 1 in. in diameter?

10. The cost of carpeting a room, 15 feet broad, with carpet $\frac{5}{8}$yd wide, at 1 R. 4a. a yard, is found to be 90 Rs. Find the length of the room.

11. A rectangular court-yard 160 cubits long and 80 cubits broad, has a walk within it along its border, 4 cubits wide. Find the cost of paving the walk with Chunar stone at 9a. 6p. per sq. foot.

12. What time will a railway train, 180 yards long, and moving at the rate of 10 miles an hour, take in passing over a bridge 40 yards long?

13. What is the solid content of an Imperial Gallon? How many imperial gallons does a cistern contain, which is 4 feet long 3 feet broad, and 2 feet deep?

14. The interior of a barrack, 200 feet long and 20 feet broad, accommodates 50 soldiers, giving 1000 cub. feet of air to each. Find the height of the barrack from the floor to the ceiling.

15. A hall is 24 feet long and 18 ft. broad. How many chairs, each 1 cubit long and 1 cubit broad, will it contain, if arranged in rows along its length, so as to leave an open space of 2 cubits in the middle, and a passage 1 cubit wide in front of each row of chairs?

16. How many bricks, each 10 inches long, 5 inches broad, and 3 inches thick, will be required for building a wall 10 feet long, 5 feet high, and 2 feet thick; and what will be the cost of the bricks at 9 Rs. 8a, per 1000?

17. How many beams each 12 feet long and measuring 7 inches by 5 inches at the end, can you get out of a piece of timber 36 feet long and measuring 2 feet 11 inches by 1 foot 9 inches at its end?

18. Find the volume of a cube whose edge is 3 feet 6 inches.

19. In a railway train, there is a certain number of passengers in the first class, twice as many in the second class, and six times as many in the third. Each passenger in the first class has to pay 1a. 6p. per mile, each in the second class, 9p. per mile, and each in the third, 3p. per mile. How many passengers are there in each class, supposing the sum total of the fares to amount to 9 Rs. a mile?

20. A square whose side is 240 yards, has a road 40 feet broad along its sides. What is the cost of repairing the road, at 3 Rs. 4a. per 100 sq. feet?

CHAPTER VI.

PRACTICE.

209. DEF. PRACTICE is a short method of finding by means of aliquot parts the value of things from the given price of a unit of any denomination. It is called SIMPLE or COMPOUND according as the quantity of the things is or is not given wholly in the denomination of the unit.

Examples of Practice are worked out in the following manner.

I. SIMPLE PRACTICE.

Ex. 1. Find the value of 372 things at 12*s.* $7\frac{1}{2}$*d.* each.
Supposing each thing to be worth £1,
the value = £372.

		£.	s.	d.
∴ the value at 10*s.* each = $\frac{1}{2}$ of £372		= 186 .	0 .	0
................2*s*....... = $\frac{1}{5}$ of value at 10*s.*		= 37 .	4 .	0
................6*d.*....... = $\frac{1}{4}$...............2*s.*		= 9 .	6 .	0
..............$1\frac{1}{2}$*d.*....... = $\frac{1}{4}$...............6*d.*		= 2 .	6 .	6
∴ value at 12*s.* $7\frac{1}{2}$*d.*		= 234 .	16 .	6

The operation is usually written thus :—

	£.	s.	d.	
10*s.* = $\frac{1}{2}$ of £1	372 .	0 .	0 .	= value at £1 each
2*s.* = $\frac{1}{5}$ of 10*s.*	186 .	0 .	0 .	= ....10*s.*......
6*d.* = $\frac{1}{4}$ of 2*s.*	37 .	4 .	0 .	= ....2*s.*......
$1\frac{1}{2}$*d.* = $\frac{1}{4}$ of 6*d.*	9 .	6 .	0 .	= ....6*d.*.....
	2 .	6 .	6 .	= ....$1\frac{1}{2}$*d.*....
	£234 .	16 .	6 .	= ....12*s.* $7\frac{1}{2}$*d.*

Ex. 2. Find the cost of 325 things at 9 a. 2 pice each?

	Rs.	a.	p.	
8 a. = $\frac{1}{2}$ of 1 R.	325	0	0	= cost at 1 R. each
1 a. = $\frac{1}{8}$ of 8 a.	162	8	0	= ....8 a.........
2 pice = $\frac{1}{2}$ of 1 a.	20	5	0	= ....1a.........
	10	2	6	= ....2pice......
	Rs. 192	15	6	= ....9 a. 2 pice

II. Compound Practice.

Ex. 1. Find the value of 18 cwt. 2 qrs. 16 lbs. of sugar at £2. 8*s*. 6*d*. per cwt.

	£	s.	d.
The value of 1 cwt. being £2. 8*s*. 6*d*., the value of 18 cwt.	=43	13	0
the value of 2 qrs. = $\frac{1}{2}$ the value of 1 cwt.	= 1	4	3
................14 lbs. = $\frac{1}{4}$..............2 qrs.	= 0	6	$0\frac{3}{4}$
................2 lbs. = $\frac{1}{7}$ ....14 lbs.	= 0	0	$10\frac{1}{2}\frac{1}{8}$
∴....................18 cwts. 2 qrs. 16 lbs.	=£45.	4*s*.	$2\frac{1}{7}$*d*.

The operation is usually written thus :—

	£	s.	d.	
	2.	8.	6	= value of 1 cwt.
			18	
2 qrs. cwt.	43.	13.	0	= ....18 cwt.
14 lbs. $\frac{1}{4}$ of 2 qrs.	1.	4.	3	= .... 2 qrs.
2 lbs.	0.	6.	$0\frac{3}{4}$	= ....14 lbs.
	0.	0.	$10\frac{1}{2}\frac{1}{8}$	.... 2 lbs.
	£45.	4.	$2\frac{1}{7}$	= value of 18 cwt. 2 qrs. 16 lbs.

Ex. 2. Find the value of 17 mds. 28 seers 3 powas at 3 Rs. 12 a. per maund..

	Rs.	a.	p.	
	3.	12.	0	= value of 1 md.
			17	
20 seers = $\frac{1}{2}$ md.	63.	12.	0	= ....17 mds.
8 seers = $\frac{1}{5}$ md.	1.	14.	0	= ....20 seers.
2 powas = $\frac{1}{16}$ of 8 seers	0.	12.	0	= .... 8 seers
1 powa = $\frac{1}{2}$ of 2 powas	0.	0.	9	= ....2 powas
	0.	0.	$4\frac{1}{2}$	= ....1 powa
	66.	7.	$1\frac{1}{2}$	= ....17 mds. 28 seers 3 powas.

210. From the above Examples it will be seen that the operation will be simplified by the selection of the most convenient aliquot parts.

We shall here subjoin a Table of aliquot parts of some of the ordinary concrete units, English and Indian; and the student will do well to commit it to memory.

TABLE OF ALIQUOT PARTS

of 1 Rupee.

$\frac{1}{2}$ = 8 a.
$\frac{1}{3}$ = 5 a. 6 gandas 2 cowries 2 krants
= 5 a. 4 p.
$\frac{1}{4}$ = 4 a.
$\frac{1}{5}$ = 3 a. 4 gandas
$\frac{1}{6}$ = 2 a. 13 gandas 1 cowry 1 krant
= 2 a. 8 p.
$\frac{1}{8}$ = 2 a.
$\frac{1}{9}$ = 1 a. 15 gandas 2 cowries 2 dantis
$\frac{1}{10}$ = 1 a. 12 gandas.
$\frac{1}{12}$ = 1 a. 4 p.
$\frac{1}{16}$ = 1 a.

of 1 anna.

$\frac{1}{2}$ = 2 pice.
$\frac{1}{3}$ = 4 pies
= 6 gandas. 2 cowries 2 krants.
$\frac{1}{4}$ = 1 pice.
$\frac{1}{5}$ = 4 gandas.
$\frac{1}{6}$ = 2 pies.
$\frac{1}{8}$ = 2 gandas 2 cowries.
$\frac{1}{10}$ = 2 gandas.
$\frac{1}{12}$ = 1 pie.

Of 1 maund.

$\frac{1}{2}$ = 20 seers.
$\frac{1}{4}$ = 10 seers.
$\frac{1}{5}$ = 8 seers.
$\frac{1}{8}$ = 5 seers.
$\frac{1}{10}$ = 4 seers.
$\frac{1}{16}$ = $2\frac{1}{2}$ seer.
$\frac{1}{20}$ = 2 seers.

of 1 seer.

$\frac{1}{2}$ = 2 powas
$\frac{1}{4}$ = 1 powa.
$\frac{1}{8}$ = 2 cht.
$\frac{1}{16}$ = 1 cht.

of £1.

$\frac{1}{2}$ = 10s.
$\frac{1}{3}$ = 6s. 8d.
$\frac{1}{4}$ = 5s.
$\frac{1}{5}$ = 4s.
$\frac{1}{6}$ = 3s. 4d.
$\frac{1}{8}$ = 2s. 6d.
$\frac{1}{10}$ = 2s.
$\frac{1}{12}$ = 1s. 8d.

of 1s.

$\frac{1}{2}$ = 6d.
$\frac{1}{3}$ = 4d.
$\frac{1}{4}$ = 3d.
$\frac{1}{6}$ = 2d.
$\frac{1}{8}$ = $1\frac{1}{2}$d.
$\frac{1}{12}$ = 1d.

of 1cwt.

$\frac{1}{2}$ = 2 qrs.
$\frac{1}{4}$ = 1 qr.
$\frac{1}{7}$ = 16 lbs.
$\frac{1}{8}$ = 14 lbs.
$\frac{1}{14}$ = 8 lbs.

of 1qr.

$\frac{1}{2}$ = 14 lbs.
$\frac{1}{4}$ = 7 lbs.
$\frac{1}{7}$ = 4 lbs.
$\frac{1}{8}$ = $3\frac{1}{2}$lbs.

Examples like the following may be worked out in a way similar to the method of Practice.

Ex. 1. Find the cost of 6 seers at 2 Rs. 12 a. per maund.

The cost of 5 seers $= \frac{1}{8}$ of 2Rs. 12a. $=$ 5 a. 2 pice;
$\therefore$ ....1 seer $= \frac{1}{5}$ of 5 a. 2 pice $=$ $4\frac{2}{5}$ pice;
and $\therefore$ ....6 seers $=$ 6 a. $2\frac{2}{5}$ pice.

Ex. 2. A man gets 7 Rs. a month; what does he get per day, supposing a month to contain 30 days?

The amount reqd. $= \frac{1}{30}$ of 7 Rs. $= \frac{1}{10}$ of $\frac{1}{3}$ of 7Rs.
Now $\frac{1}{3}$ of 7 Rs. $= \frac{1}{3}$ of 6 Rs. $+ \frac{1}{3}$ of 1 R.
$=$ 2 Rs. $+$ 5a. 4 pies;
$\therefore \frac{1}{30}$ of 7 Rs. $= \frac{1}{10}$ of 2 Rs. 5 a. 4 pies.
$=$ 3 a. $8\frac{4}{5}$ pies.

Ex. 3, A man supplies milk for a month of 31 days at 3 seers a day. What will be his charge, if milk sells at 8 seers a rupee?

The quantity of milk supplied
$= 3 \times 31 = 93$ seers.

	Rs.	a.	p.
The value of 88 seers =	11	0	0
...............5........ =	0	10	0
$\therefore$ ....93...... =	11	10	0

Ex. XXXIX.

Find the value of

1. 50 things at 2 Rs. 2 a. each, and 64 things at 3 Rs. 5 a. each.
2. 72 things at £1. 10*s*. each, and 55 things at 15*s*. 6*d*. each.
3. 126 things at 13*s*. 4*d*. each, and 100 things at £2. 5*s*. each.
4. 30 things at 2 Rs. 6 a. each, and 31 things at 3 a, 6 p. each.
5. 40 things at 2 Rs. 5 a. each, and 120 things at 5 a. 6 p. each.
6. 80 things at 3 Rs. 7 a. each, and 90 things at 2 Rs. 9 a. each.

7. 720 things at £3. 6*s*. each, and 885 things at 9*s*. 6*d*. each.

8. 144 things at £19. 17*s*. each, and 288 things at £7. 6*s*. 5*d*. each.

9. 1000 things at £15. 16*s*. 11*d*. each, and 925 things at £5. 10*s*. each.

10. 1000 things at 2 Rs. 5 a. each, and 1500 things at 3 Rs. 7 a. each.

11. 1285 things at 5 Rs. 4 a. each, and 725 things at 7 Rs. 6 a. 4 p. each.

12. 361 things at 16 Rs. 15 a. each, and 2500 things at 9 Rs. 6 a. each.

13. 15 cwt. 2 qrs. 10 lbs. at 2*s*. 6*d*. per lb.

14. 17 cwt. 3 qrs. 14 lbs. at £2. 9*s*. per cwt.

15. 110 cwt. 2 qrs. 20 lbs. at £1. 13*s*. 4*d*. per cwt.

16. 7 mds. 35 seers of sugar at 13 Rs. 2 a. per maund.

17. 225 mds. 33 seers of rice at 4 Rs. 14 a. 6 p. per maund.

18. 72 mds. 25 seers of rice at 5 Rs. 10 a. per maund.

19. 16 yds. of silk at £1. 3*s*. per yard.

20. 170 yds. of linen at 2*s*. 6*d*. per yard.

21. 8 bghs. 11 kths. of land at 65 Rs. 8 a. per katha.

22. 13 bghs. 7 kths. of land at 49 Rs. 8 a. per katha.

23. 15 bghs. 16 kths. of land at 125 Rs. per katha.

24. 140 ft. 6 in. at 1 R. 4 a. 6 p. per foot.

CHAPTER VII.

PROPORTION AND VARIATION. RULE OF THREE.

SECTION I. PROPORTION AND VARIATION.

211. DEFS. RATIO is the relation which one number bears to another in respect of magnitude, the comparison being made by considering how many times or parts of a time the latter is contained in the former.

The two numbers are called the TERMS of the ratio, the former being called the ANTECEDENT, and the latter the CONSEQUENT.

A ratio is written by writing its terms one after the other and placing a colon (:) between them. Thus the ratio of 3 to 4 is written,

$$3 : 4.$$

It is evident from the definition, that the same ratio may also be represented by the fraction $\frac{3}{4}$; for by the definition, the ratio of 3:4 denotes the number of times or rather parts of a time that 4 is contained in 3, and the fraction $\frac{3}{4}$ also denotes the same thing (Art. 84).

212. The terms of a ratio must either be both abstract numbers or both concrete numbers of the *same* kind; for otherwise there can be no comparison between them; thus we cannot compare 3 feet with 4 hours in respect of magnitude: and the ratio itself, as it indicates the *number* of *times* or *parts* of a time that the antecedent contains the consequent, must always be an abstract number.

Again the ratio of one concrete number to another, when both are expressed in the *same denomination*, will be the same as the ratio of the former number to the latter, both being regarded as abstract numbers. Thus, the ratio of 3 feet to 4 feet is the same as the ratio of the abstract number 3 to the abstract number 4, and is expressed by the fraction $\frac{3}{4}$. But the ratio of 3 feet to 4 inches, that is of 36 inches to 4 inches, will not be the same as that of 3 to 4, but will be the same as that of 36 to 4, and will be expressed not as $\frac{3}{4}$ but as $\frac{36}{4}$.

213. Def. Four numbers are said to be Proportionals or to constitute a Proportion, when the ratio of the first to the second is equal to that of the third to the fourth, or in other words, when the first contains the second as often as the third contains the fourth.

A proportion is written by writing its ratios one after the other with a double colon (: :) between them. Thus, taking any four numbers that are proportional, for example 3, 4, 12, 16, the proportion is written thus :—

$$3 : 4 : : 12 : 16 ;$$

and it is read thus :—

3 is to 4 as 12 to 16.

. If any four numbers 3, 4 , 12, 16 constitute a proportion, then $\frac{3}{4} = \frac{12}{16}$.

For $\frac{3}{4}$ denotes the number of parts of a time that 3 contains 4 and $\frac{12}{16}$ denotes the number of parts of a time that 12 contains 16 ; and these are equal by the definition of proportion.

214. In the preceding Article, we have seen four numbers constituting a proportion ; but we may also have three numbers constituting a proportion, and then they must be such that the first is to the second as the second is to the third.

Thus the three numbers 3, 6 and 12 constitute the proportion 3 : 6 : : 6 : 12,

and $\frac{3}{6} = \frac{6}{12}$.

215. The terms of each of the two ratios constituting a proportion must satisfy the conditions mentioned in Art. 212. But it is not necessary that all the four terms should be simultaneously abstract numbers, or simultaneously concrete numbers. The terms of one ratio may be abstract numbers, and those of the other, concrete numbers. The ratio of two concrete numbers, which is always an abstract number, may be equal to that of two abstract numbers, or of two concrete numbers of another kind.

Thus 2Rs. : 3Rs. :: 12ft. : 18ft.

For $\frac{2\text{Rs.}}{3\text{Rs.}}$ = the abstract fraction $\frac{2}{3}$.

Similarly $\frac{12\text{ft.}}{18\text{ft.}} = \frac{12}{18} = \frac{2}{3}$.

216. PROP. I. If four numbers are proportionals when taken in a certain order, they are also proportionals when taken in the reverse order.

Thus, taking for example the numbers 3, 5, 9, 15, which constitute the proportion $3 : 5 :: 9 : 15$ so that $\frac{3}{5} = \frac{9}{15}$,

we have $1 \div \frac{3}{5} = 1 \div \frac{9}{15}$,

or $\frac{5}{3} = \frac{15}{9}$,

or $5 : 3 :: 15 : 9$, or $15 : 9 :: 5 : 3$.

PROP. II. When four numbers are proportionals, the product of the extremes = the product of the means.

For taking the same proportion

$$3 : 5 :: 9 : 15,$$

we have $\frac{3}{5} = \frac{9}{15}$;

or multiplying both by 5×15 we have

$\frac{3}{5} \times 5 \times 15 = \frac{9}{15} \times 5 \times 15$, or $3 \times 15 = 5 \times 9$.

217. DEF. When four numbers are proportionals, the fourth is said to be a FOURTH PROPORTIONAL to the other three.

When three numbers are proportionals, the third is said to be a THIRD PROPORTIONAL to the other two, and the second, a MEAN PROPORTIONAL between the other two.

The process for finding a fourth proportional to three numbers, or a third proportional to two numbers, consists merely in the application of Prop. II of the preceding Article; and is usually stated in the manner given below.

Ex. 1. Find a fourth proportional to 7, 9 and 21.

Let x be the fourth proportional required.

Then $7 : 9 :: 21 : x$,

$\therefore \quad 7 \times x = 9 \times 21$, or dividing by 7,

$$x = \frac{9 \times 21}{7} = 27.$$

Ex. 2. Find a third proportional to 7 and 9.

Let x be the third proportional required.

Then $7 : 9 :: 9 : x$,

$\therefore \quad 7 \times x = 9 \times 9$, or dividing by 7,

$$x = \frac{9 \times 9}{7} = 11\tfrac{4}{7}.$$

218. DEFS. One quantity is said to VARY DIRECTLY as another, when either of them being *increased* or *decreased*, the other is *increased* or *decreased* in the same proportion.

Thus, at a giving rate of walking, say 2 miles an hour, the distance walked may be said to vary directly as the time; since in any time, say 3 hours, the distance walked is 6 miles, and increasing the time to 4 hours, the distance is increased to 8 miles, so that

the former time 3 hours : the increased time 4 hours
: : the former distance 6 miles : the increased distance 8 miles.

Hence, it is evident, that when one quantity varies directly as another, any two numerical values of the former expressed in the same denomination, and the corresponding numerical values of the latter expressed in the same denomination, and *taken in the same order*, will constitute a proportion.

One quantity is said to VARY INVERSELY as another, when either of them being *increased* or *decreased*, the other is *decreased* or *increased* in the same proportion.

Thus in walking a given distance, say 24 miles, the time of walking may be said to vary inversely as the rate, since for any rate, say 2 miles an hour, the time will be 12 hours, and *increasing* the rate to 3 miles an hour, the time is *decreased* to 8 hours, so that

the former rate 2 miles : the increased rate 3 miles
: : the latter or decreased time 8 hours : the former time 12 hours.

Hence it is evident that when one quantity varies inversely as another, any two numerical values of the former expressed in the same denomination, and the two corresponding numerical values of the latter expressed in the same denomination, and *taken in the reverse order*, will constitute a proportion.

219. When one quantity varies directly or inversely as another, and two values of the former and only one corresponding value of the latter are given, the other corresponding value of the latter can be determined as in the following Examples.

Ex. 1. If 6 yds of cloth cost 13 Rs. 8a., find the price of 9 yds.

Let $x=$ the price of 9 yds. in rupees.

Then $\because$ 13 Rs. 8a. $=13\frac{1}{2}$ Rs. $=\frac{27}{2}$ Rs.,

and $\because$ the price of cloth varies directly as the quantity of cloth, we have by Art. 218.

$6 : 9 :: \frac{27}{2} : x,$

$\therefore 6 \times x = 9 \times \frac{27}{2}$

and $\therefore x = \frac{9 \times 27}{2 \times 6} = \frac{81}{4} = 20\frac{1}{4}.$

$\therefore$ price reqd. = 20 Rs. 4 a.

Ex. 2. If 6 men can do a piece of work in 9 days, in how many days will 15 men do the same work?

Here $\because$ the larger the no. of men employed, the shorter becomes the time for finishing the work, $\therefore$ the number of men varies inversely as the time.

Now let x = the no. of days reqd.

Then $6 : 15 :: x : 9$;

$\therefore 15 \times x = 6 \times 9,$

or $x = \frac{6 \times 9}{15} = \frac{18}{5} = 3\frac{3}{5}$, the no. of days required.

Ex. 3. If 9 men working 6 hours a day, can do a piece of work in 12 days, in how many days will 8 men do the same work, working 9 hours a day?

Let x = the no. of days reqd.

Then the time taken in doing the work in the one case $= 6 \times 12$ hours,

and in the other case $= 9 \times x$ hours.

Now, as in Ex. 2, the no. of men varies inversely as the time;

$\therefore$ the proportion will stand thus :—

$9 : 8 :: 9 \times x : 6 \times 12,$

and $\therefore 8 \times 9 \times x = 9 \times 6 \times 12$

and $\therefore x = \frac{9 \times 6 \times 12}{9 \times 8} = 9$, the no. of days required.

Ex. 4. If 7 men can reap 18 bghs. in 12 hrs., how many men will be able to reap 45 bghs. in 14 hrs?

Let x = the no. of men reqd.

Then in the former case we have

7 men working 12 hours,

which is the same as

7×12 men working 1 hour,

and the work done is the reaping of 18 bghs.

In the latter case, we have
x men working 14 hours,
which is the same as
$x \times 14$ men working 1 hour,
and the work done is the reaping of 45 bghs.

Now in the given time, 1 hour, the no. of bighas reaped will vary directly as the no. of men;

$$\therefore\ 18 : 45 :: 7 \times 12 : x \times 14;$$
$$\therefore\ 18 \times x \times 14 = 45 \times 7 \times 12$$
and $\therefore$
$$x = \frac{45 \times 7 \times 12}{18 \times 14} = 15.$$

220. We may here notice some of the ordinary instances in which one quantity varies as another.

I. With a given area, the length varies inversely as the breadth.

II. With a given length, the breadth varies directly as the area.

III. With a given rate of motion, the distance passed varies directly as the time, supposing the rate to be uniform.

IV. With a given time, the distance passed varies directly as the rate of motion, on the same supposition.

V. With a given distance, the rate of motion varies inversely as the time on the same supposition.

VI. With a given time, the work done varies directly as the agency.

VII. With a given agency, the work done varies directly as the time.

VIII. With a given work, the agency varies inversely as the time.

IX. With a given rate of price, the price paid varies directly as the quantity of article purchased.

X. With a given price, the quantity of article purchased varies inversely as the rate of price.

The truth of the above statements will be made evident by one or two examples.

SECTION II. RULE OF THREE.

221. In Art. 219, we have already seen that when three quantities are given we can sometimes find out a fourth. The method of finding out the fourth is called the Rule of Three. It may be thus defined :—

DEFS. The RULE OF THREE is a method by which of four quantities which are proportionals, any three being given, the fourth can be determined.

It is called DIRECT or INVERSE according as the variation upon which it depends is direct or inverse. Thus Ex. 1 of Art. 219 is an example of Rule of Three Direct ; Ex. 2, one of Rule of Three Inverse. It is called SINGLE or DOUBLE according as only three or more than three quantities are given from which the required one is to be ascertained. Examples 3 and 4 of Art. 219 are examples of Double Rule of Three.

It is called the Rule of Three because there are *three* quantities given from which we find the fourth or required quantity ; and from its application to the solution of a large and important class of questions in the common affairs of life, it has sometimes been called the Golden Rule.

222. The Rule for working out Examples of Rule of Three may be gathered from Art. 219. We can state it thus :—

RULE. Put x for the required number or the required quantity expressed numerically. Make the necessary reductions to the same denomination. Then from the nature of the question ascertain what the numbers are that are proportionals, and state the proportion. Put the product of the extremes equal to the product of the means, and then divide the product which does not contain x by the product of all the factors of the other product except x. The quotient will be the required number or the required quantity expressed numerically.

The above Rule is quite general, and will apply to all cases of Rule of Three, whether Direct or Inverse, Single or Double, as will be seen from the Examples given below.

I. SINGLE RULE OF THREE.

Ex. 1. Find the price of 5 yds. 3 in. of silk when 3 yds. cost 6 Rs. 12 a.

Let x = the price reqd. in rupees.

Then ∵ the quantity of silk in one case = 3 yds.

and the quantity of silk in the other case = 5 yds. 3 in. $= 5\frac{1}{4}$ yds.;

and the price in the former case = 6 Rs. 12 a. $= 6\frac{3}{4}$ Rs.,

and the price in the latter case $= x$ Rs.;

and ∵ the price varies directly as the quantity of the article;

$$\text{we have } 3 : 5\tfrac{1}{4} :: 6\tfrac{3}{4} : x,$$

$$\therefore 3 \times x = 5\tfrac{1}{4} \times 6\tfrac{3}{4} = \tfrac{27}{4} \times \tfrac{21}{4},$$

$$\text{and } x = \frac{21 \times 27}{3 \times 4 \times 4} = \frac{189}{16} = 11\frac{13}{16};$$

∴ *the price reqd.* $= 11\frac{13}{16}$ Rs. = 11 Rs. 13 a.

Ex. 2. *If 7 men can reap a field 6 bghs. in length and 3 bghs. in breadth in 14 hrs., in how many hours will 8 men be able to reap the same field?*

Let x = the no. of hrs. reqd.

Then because the work, *viz.*, the reaping of the field is the same in both cases, the no. of men will vary inversely as the time; and

$$\therefore 7 : 8 :: x : 14;$$

$$\therefore 8 \times x = 7 \times 14, \text{ or } x = \frac{7 \times 14}{8} = \frac{49}{4} = 12\tfrac{1}{4}.$$

Here it will be seen that the quantities 6 bghs. and 3 bghs. are superfluous, as being the same in both cases, they do not affect the question.

Ex. 3. If 7 men can reap a field 6 bghs. in length and 3 bghs. in breadth in 14 hrs., what is the area of the field that they can reap in 18 hrs.?

Let x = the area reqd. in sq. bighas.

The area in the former case $= 3 \times 6$ or 18 sq. bghs.

Now ∵ the work done is measured by the no. of bighas reaped,

and ∵ the no. of men working is the same in both cases, the time will vary directly as the no. of bighas;

$$\text{and } \therefore 14 : 18 :: 18 : x;$$

$$\therefore x = \frac{18 \times 18}{14} = \frac{162}{7} = 23\frac{1}{7};$$

and ∴ the area reqd. $= 23\frac{1}{7}$ bghs. = 23 bghs. $2\frac{6}{7}$ kths.

Here it will be seen that the number of men being the same in both cases does not affect the question.

Ex. 4. In the preceding Example, what will be the length of the field in the second case if its breadth is 9 bighas?

Let x = reqd. length in bighas.
Then the area in the second case

$$= 9 \times x \text{ bighas}$$

and ∴ the proportion will stand thus:—

$$14 : 18 :: 18 : 9 \times x;$$

$$\therefore 9 \times x \times 14 = 18 \times 18,$$

$$\therefore \quad x = \frac{18 \times 18}{9 \times 14} = \frac{18}{7} = 2\frac{4}{7},$$

and ∴ the length reqd. $= 2\frac{4}{7}$ bghs. = 2 bghs. $11\frac{3}{7}$ kths.

Ex. 5. If 3 yds. of silk cost as much as 7 yds. of linen, how many yards of silk can be given in exchange for 42 yds. of linen?

Let x = the no. of yds. reqd.

Then x must bear the same relation to 42 in respect of magnitude that the number 3 does to the number 7; in other words, the ratio of x to 42 = that of 3 to 7;

$$\therefore x : 42 :: 3 : 7,$$

and ∴ $x \times 7 = 42 \times 3$ or $x = \frac{42 \times 3}{7} = 18.$

∴ 18 is the no. of yds. of silk reqd.

Ex. 6. A watch is set right at 1 o'clock P. M. on Monday, and at 9 o'clock P. M. on Tuesday it is found to be 3′ too fast. Supposing its rate regular, what will be the time by the watch at 6 o'clock A. M. on Saturday?

From 1 P. M. Monday to 9 P. M. Tuesday there are 24 + 8 or 32 hrs. and in that time the watch gains 3′.

From 1 P. M. Monday to 6 A. M. Saturday there are $4 \times 24 + 17$ or 113 hrs., and in that time let the watch gain x'.

Then ∵ the rate is regular, the gain by the watch will vary directly as the time;

$$\therefore 32 : 113 :: 3 : x,$$

and ∴ $32 \times x = 3 \times 113$ or $x = \frac{3 \times 113}{32} = \frac{339}{32} = 10\frac{19}{32}.$

$\therefore$ the watch has gained $10\frac{19}{32}'$ and the time by the watch is $10\frac{19}{32}'$ past 6 o'clock A. M.

II. DOUBLE RULE OF THREE.

Ex. 7. If 8 men can do a piece of work in 18 days, working 7 hours a day, how many hours a day must 12 men work to finish in 21 days a piece of work twice as great?

To finish the second piece of work, 8 men must work for 2×18 or 36 days of 7 working hours each.

Now let $x =$ the no. of hours reqd.

Then the second piece of work is done

by 8 men in 36×7 hrs.,

and by 12 men in $21 \times x$ hrs.

and as the no. of men must vary inversely as the time,

$$\therefore\ 8 : 12 :: 21 \times x : 36 \times 7;$$

$$\therefore\ 12 \times 21 \times x = 8 \times 36 \times 7$$

and $\therefore$ $$x = \frac{8 \times 36 \times 7}{12 \times 21} = 8,$$ the no. of hrs. reqd.

In this Example, we have three things involved, workmen, work done, and time, and all three are different in the two cases.

In the process given above, we have made the work done the same in both cases, by multiplying the time in the former case by 2; and thus the question is reduced to one in which the work done remains the same, and the time and the number of workmen are the only varying elements, and then as we know that the former varies inversely as the latter, we state the proportion accordingly. We might have worked out the Example by reducing the time to the same duration in the two cases by altering the number of workmen, and then having the number of workmen and work done for our varying elements. Thus,

8 men working 18×7 hours, is the same as $8 \times 18 \times 7$ men working 1 hour;

Similarly 12 men working $21 \times x$ hours is the same as $12 \times 21 \times x$ men working 1 hour;

and in the former case the work done being 1, in the latter it is 2;

and as in the given time 1 hour, the work done must vary directly as the number of men,

$$\therefore 8 \times 18 \times 7 : 12 \times 21 \times x :: 1 : 2;$$

and $\therefore 12 \times 21 \times x \times 1 = 2 \times 8 \times 18 \times 7$,

$$\text{whence} \quad x = \frac{2 \times 8 \times 18 \times 7}{12 \times 21} = 8.$$

In working out Examples of Double Rule of Three in which three varying elements occur, the chief art consists in reducing one of the three elements to a constant quantity; and although the reduction of any one of the elements will give us a proportion from which the required quantity may be found, we must try in every case to make use of the most convenient reduction.

Ex. 8. If 9 masons, in 10 days of 8 working hours each, can build a wall 48 ft. long, 10 ft. high, and 2 ft. deep, how many masons will be required to build a wall 60 ft. long, 12 ft. high, and 3 ft. deep, in the same number of days, but working only 6 hours daily?

Let x = the number of masons reqd.

Then in the former case we have

9 masons working for 10×8 hours,

which is the same as

$9 \times 10 \times 8$ masons working for 1 hour;

and in the latter case we have

x masons working for 10×6 hours,

which is the same as

$x \times 10 \times 6$ masons working for 1 hour.

Now in the former case,

the work done is measured by $2 \times 10 \times 48$ cub. ft.,

and in the latter case,

it is measured by $3 \times 12 \times 60$ cub. ft.;

and $\because$ in the given time 1 hour,

the work done will vary directly as the number of masons,

$$\therefore 2 \times 10 \times 48 : 3 \times 12 \times 60 :: 9 \times 10 \times 8 : x \times 10 \times 6;$$

and $\therefore 2 \times 10 \times 48 \times x \times 10 \times 6 = 3 \times 12 \times 60 \times 9 \times 10 \times 8$,

$$\text{whence } x = \frac{3 \times 12 \times 60 \times 9 \times 10 \times 8}{2 \times 10 \times 48 \times 10 \times 6} = 3 \times 9$$

$= 27$, the no. of masons reqd.

In the foregoing process we have reduced the time to the same duration 1 hour in both cases. We can work out the Example by reducing the number of men to a constant thus :—

9 men working 10×8 hrs. is the same as
1 man working $9 \times 10 \times 8$ hrs. ;
so, x men........10×6 hrs. is the same as
1 man........... $x \times 10 \times 6$ hrs.

Now with the same 1 workman
the work done will vary directly as the time ;
$\therefore 2 \times 10 \times 48 : 3 \times 12 \times 60 :: 9 \times 10 \times 8 : x \times 10 \times 6$;
and $\therefore 2 \times 10 \times 48 \times x \times 10 \times 6 = 3 \times 12 \times 60 \times 9 \times 10 \times 8$,
whence $x = 27$.

Ex. 9. If the carriage of 25 mds. over 50 miles cost 18 Rs. 12 a., over how many miles can 30 mds. be carried for 20 Rs., supposing there to be a uniform rate for every 1 maund carried over 1 mile?

Let x = the no. of miles reqd.
Then, the unit being 1 md. carried 1 mile.
the work in the former case
$= 25 \times 50$ units,
and the work in the latter case
$= 30 \times x$;
and the work will vary directly as its cost ;
$\therefore 25 \times 50 : 30 \times x :: 18\frac{3}{4} : 20$;
$\therefore 30 \times x \times \frac{75}{4} = 25 \times 50 \times 20$,

$$\text{or } x = \frac{25 \times 50 \times 20 \times 4}{30 \times 75} = \frac{400}{9}$$
$$= 44\frac{4}{9}, \text{ the no. of miles reqd.}$$

Ex. 10. There is just sufficient rice in store to feed 50 men for 30 days, giving 15 chataks to each man daily. To how many chataks must the daily allowance be reduced, to feed for 40 days these and 10 men more without any addition to the store?

Let x = the no. of chataks reqd. Then to feed 50 men for 30 days will require as much food as to feed 30×50 men for 1 day ; and so to feed $(50 + 10)$ or 60 men for 40 days is the same as to feed 40×60 men for 1 day.

Now in 1 day, with a given store, the allowance to each man must vary inversely as the no. of men ;

$\therefore 30 \times 50 : 40 \times 60 :: x : 15$

and $\therefore x \times 40 \times 60 = 30 \times 50 \times 15$

or $$x = \frac{30 \times 50 \times 15}{40 \times 60} = \frac{75}{8}$$

$= 9\frac{3}{8}$, the no. of chataks reqd.

This Example may also be worked thus :—

Quantity consumed in the former case

$= 50 \times 30 \times 15$ chts.,

quantity consumed in the latter case

$= 60 \times 40 \times x$ chts. ;

and these two quantities must evidently be equal ;

$\therefore 50 \times 30 \times 15 = 60 \times 40 \times x$,

whence $x = 9\frac{3}{8}$.

Ex. 11. If 7 seers of bread cost 2 Rs. when flour is 5Rs. a maund, what is the price of flour per maund, when 8 seers of bread cost 3 Rs., supposing the price of bread to vary directly as the price of flour ?

Let $x =$ the price reqd. in rupees.

Now in the former case, we get 7 seers of bread for 2 Rs.

i. e. the price of 1 seer is $\frac{2}{7}$R.

Similarly, in the latter case,

the price of 1 seer is $\frac{3}{8}$ R.

And as the price of bread varies directly as the price of flour,

$\therefore \frac{2}{7} : \frac{3}{8} :: 5 : x$,

and $\therefore x \times \frac{2}{7} = 5 \times \frac{3}{8}$;

whence $$x = \frac{5 \times 3 \times 7}{8 \times 2} = \frac{105}{16} = 6\frac{9}{16},$$

and $\therefore$ the reqd. price of flour per md. = 6Rs. 9a.

Ex. 12. If 5 men in 12 days earn £2, in how many days will 6 men earn £5 ?

Let $x =$ the number of days reqd.

Now the earnings of 5 men in 12 days will be the same as the earnings of 12×5 men in 1 day ;

and so the earnings of 6 men in x days will be the same as the earnings of $x \times 6$ men in 1 day.

And as in the given time 1 day, the earnings will vary directly as the number of men,

$\therefore 2 : 5 :: 12 \times 5 : x \times 6$;

$\therefore 2 \times x \times 6 = 5 \times 12 \times 5$

and $\therefore x = \dfrac{5 \times 12 \times 5}{2 \times 6} = 25$, the no. of days reqd.

223. The applicability of the Rule of Three to the solution of any question, depends upon the subsistence of the relation of proportion among the quantities involved ; and when this relation does not exist, the Rule of Three will not apply. Thus in the question,

"If 8 mangoes are worth 1 R. what is the price of 15 pineapples?"

as evidently, the relation of proportion does not subsist among the quantities involved, *viz.*, the number of mangoes, the number of pineapples, the price given, and the price required, the Rule of Three will not apply.

But if the question be this,

"If 8 mangoes are worth 1 R., what is the price of 15 pineapples, supposing 2 mangoes to be worth as much as 3 pineapples?"

then as we can reduce the 8 mangoes to their equivalent number of pineapples in value by the proportion,

$$2 : 8 :: 3 : x, \text{ whence } x = \frac{8 \times 3}{2} = 12,$$

we can state the question in other words thus:

"If 12 pineapples are worth 1R. what is the price of 15 pineapples?"

And to this clearly the Rule of Three applies. And since the price varies directly as the number of pineapples, the process as usual will stand thus:—

Let $x =$ the price reqd. in rupees.

Then $12 : 15 :: 1 : x$;

$\therefore x \times 12 = 15 \times 1$, and $\therefore x = \frac{15}{12} = \frac{5}{4} = 1\frac{1}{4}$;

and $\therefore$ the price reqd. $=$ 1R. 4a.

Again. if the question be,

"What will the East Indian Railway Company charge for carrying 1 maund over 25 miles, when they charge 12 annas for carrying the same weight over 50 miles?"

although it may appear at first sight to be an ordinary ques tion of Rule of Three, yet it may not always be really so. For. the Railway Company may charge the same amount of 12 annas for every distance below 50 miles, or it may adjust its table of fares in any other way than in proportion to the distance, and in that case the Rule of Three will not apply. It will be a question of Rule of Three on the supposition that the Railway Company charges at a uniform rate per mile; and it is only in that case that the solution will be given in the usual way, by putting $x =$ the cost of carriage reqd. in annas, and stating the proportion $50 : 25 :: 12 : x$,

whence $x = 6$. the no. of annas reqd.

Before applying, therefore, the Rule of Three to the solution of any practical question, the student should enquire whether the relation of proportion exists among the quantities involved.

In the theoretical questions set to be solved by the Rule of Three, it is always assumed either expressly or by implication that the rate involved in the question is uniform.

Thus in Example 1 of Art. 222, the. rate or price per yard is assumed to be uniform. So in Examples 7 and 8 of Art. 222, each man is supposed to work at the same uniform rate per hour

224. Sometimes again there may be the relation of proportion existing but in a peculiar way. Thus in the question,

"What is the price of a diamond weighing 9 rattis, when another diamond weighing 3 rattis costs 90 Rs., supposing the value of diamonds to vary as the square of their weight?"

assuming $x =$ the price required in rupees, the proportion will not be,

$$3 : 9 :: 90 : x,$$

but will be

$$3^2 : 9^2 :: 90 : x,$$

i. e. $9 : 81 :: 90 : x$,

whence $x = \dfrac{81 \times 90}{9} = 810$;

and the price reqd. is 810 Rs.

Ex. XL.

1. Find a fourth proportional to

(1) 1, 2, and 3.
(2) 12, 14, and 16.
(3) £6, £9, and £12.
(4) £14, £12, and 7*s.*
(5) £10, 10*s.*, and 25Rs.
(6) 2 mds. 20 srs., 1sr., and 100.
(7) 4, 5, and 5 bghs.
(8) 6Rs. 4a., 1R. 9a., and 16yds. 2ft.

2. Find a third proportional to

(1)	5 and 15.	(2)	10 and 12.
(3)	192 and 24.	(4)	1728 and 144.
(5)	£1 and 5*s*.	(6)	1R. and 1a.

3. The lengths of three poles are such that the first contains the second as often as the second contains the third; and there are as many cubits in the second as there are yards in the first. Given that the length of the first is 60 ft; find the lengths of the other two.

4. Two rectangular fields, whose areas are equal, are such that the length of the one contains as many bighas as there are kathas in the breadth of the other. How often is the breadth of the former contained in the length of the latter?

5. If 16yds. of cloth cost 15Rs. how much will 20 yds. cost?

6. If 28mds. of rice can be had for 91Rs. 7a., how much rice can be had for 52Rs. 4a.?

7. If 18mds. of sugar cost 225Rs., find the value of 22 mds. 10 seers.

8. If 16 cwt. of sugar can be had for £20. 16*s.*, how much sugar can be had for £26?

9. For a certain sum, 15 yds. of silk can be had of a certain quality, or 25 yds. of silk of an inferior quality. Compare the prices of the two kinds of silk per yard, and find the price of the latter supposing that of the former to be 10*s.* per yard.

10. If an ounce of quinine can be had for 10 Rs., what is the price of 3 drams?

11. If a tola of pure silver is worth 1R. 1a. $5\frac{5}{11}$p., and if pure gold is worth 16 times as much as pure silver, what is the value of a seer of pure gold?

12. A gentleman pays an income tax of £17. 10 *s.* a year when the tax is 7*d.* in the £. What is his annual income?

13. The assets of an insolvent debtor amount to 24000Rs., and his creditors can get only 10a. in the rupee. Find the amount of his debts.

14. The assets of an insolvent amount to 15312 Rs. 8a., and his debts amount to 35000Rs. How much can his creditors get in the rupee?

15. Find the annual value of a revenue-free estate which pays 171 Rs. 14a. annually as road-cess, such cess being levied at the rate of one-half of an anna in the rupee of the annual value.

16. The annual value of a certain revenue paying estate is twice as much as the annual revenue payable for it, and the estate pays every year as road cess the sum of 2578 Rs. 2a. Find the annual value of the estate, supposing the road cess to be payable at the rate of one-half of an anna in the rupee of the annual value, less a deduction at one-half of the said rate for every rupee of the revenue.

17. If 5 boys earn as much as 3 men, what is the monthly earning of a boy, supposing a man to earn 10 Rs. a month?

18. If the wages of 5 carpenters amount to as much as the wages of 6 masons, what will 16 carpenters earn in 1 day supposing the weekly earnings of 10 masons to be 21 Rs. 14a.?

19. If 6 men earn as much as 10 boys, how much will 15 boys earn per week, supposing a man's daily earning to be 5a.?

20. Supposing the value of diamonds to vary as the square of their weight, find the relation between the values of two diamonds weighing 3 and 5 rattis respectively.

21. The area of a circle varies as the square of its diameter. Find the area of a circle 8 ft. in diameter, supposing a circle 6 feet in diameter to contain 28·27 sq. ft.

22. The circumference of a circle varies as its radius. Find the length of the circumference when the radius is 3 ft., supposing the circumference of a circle whose radius is 2 feet 6 in., to be 15·7080 feet.

23. The circumference of a circle is to its radius as $3·1416 : \frac{1}{2}$. Find the radius of a circle whose circumference is 3 miles.

24. The area of square varies as the square of its diagonal. Find the area of a square field which measures 7 bighas along its diagonal, supposing the area of a square, whose diagonal is 5 ft., to be 12·5 square feet.

25. A rectangular plot of land measuring 4 bghs. 5 kths. by 3 bghs. 10 kths., is worth 15000 Rs. What is the value of another plot which is 5 times as long and 3 times as broad?

26. If an estate, which contains 10000 acres, yields an annual income of £25000, what would be the income of another estate which contains an area of 8 square miles, at the same rate.

27. A zemindari containing 30000 bghs. of land yields an annual income of 28000 Rs. One-third of the whole area is waste land which yields nothing; and of the remainder, one-tenth consists of mulberry land, and the rest is paddy land. Supposing the former description of land to yield per bigha five times as much rent as the latter, find the rent per bigha of each.

28. One-tenth part of the land comprised in an estate which contains 15000 bghs., is homestead land, and the rest is arable land; and the rate of rent for the former is to that for the latter as 2·5 : 1. What is the rent per bigha for each description of land, supposing the whole rent realizable from the estate to be 17250 Rs.?

29. If 3 men or 5 boys earn 6 Rs. 9 a. per week, how much will 5 men and 3 boys earn in one year? (1 year = 52 weeks.)

30. If 5 horses eat as much as 7 ponies, and if the feeding of 1 horse cost 18 rupees a month, what will be the monthly cost of feeding 3 horses and 2 ponies?

31. Divide 1600 Rs. between *A* and *B*, so that their shares may be as. 3 : 5.

32. Divide 2400 Rs. among A, B, and C, so that the shares of A and B may be as 1 : 2, and those of B and C as 2 : 5.

33. The Thakbast scale being 16 inches to the mile, how much is it to the bigha, and what is the length represented by an inch on the Thak map?

34. If the greatest length of India, which is 1800 miles, appears on a map to be 1ft. 10·5 in., what would be the length on the map of the river Ganges which is 1500 miles long?

35. Supposing the diameter of the Earth to be 8000 miles, and the height of the highest mountain upon it to be 28000ft. what decimal of an inch would represent this height on a globe 2ft. in diameter?

36. Two rectangular fields having the same length contain 140 bghs. and 240 bghs. respectively. Given that the breadth of the former is 3 bghs. 10 kths., find the breadth of the latter.

37. Two rectangular fields having the same area, measure along their lengths 300 poles and 1 mile respectively. Given that the breadth of the former is half a mile, find the breadth of the latter.

38. What must be the length of a plot of land that is 3 bghs. 15 kths. broad, in order that it may be given in exchange for a square plot measuring 4 bghs. 5 kths. along its side?

39. Two plots of land of the same length, are worth respectively 5400 Rs. and 7200 Rs. What is the breadth of the former, if that of the latter be 4 bghs. 10 kths.?

40. A tank is to be given in exchange for a plot of land of equal length. The value of a katha of the latter is twice as much as that of a katha of the former. Find the breadth of the land supposing that of the tank to be 120ft.

41. A special train moving at a uniform rate, leaves Howrah for Allahabad at 6.30′ P. M. on Wednesday, touches Burdwan at 51′ past 9 o'clock P. M. of the same day, and arrives at Allahabad at 42′ past 10 o'clock P. M. on Thursday. Given that Burdwan is 67 miles from Howrah, find the distance of Allahabad from Howrah by rail.

42. An express train proceeding at the rate of 30 miles an hour, after passing through two-thirds of its journey, meets with an accident which reduces its rate, and the train in consequence takes the same time to complete the remainder of its journey, that it took in travelling up to the place where the accident happened. Find the rate of the train after the accident.

43. An express train moving at the rate of 24 miles an hour, leaves Howrah for Allahabad which is at a distance of 564 miles; and on reaching Mokameh which is at a distance of 282 miles it meets with an accident which compels it to reduce its rate, and the train is in consequence 7 hrs. 3′ late in reaching Allahabad. Find its rate after the accident.

44. Supposing that light takes 8′ to come from the Sun to the Earth, and that the Earth takes 365 days, 6 hrs. to describe a circle round the Sun, compare the velocity of the Earth with the velocity of the light. (The circumference of a circle is to its radius as $3 \cdot 1416 : \frac{1}{2}$.)

45. A room 8 cubits broad is to be covered with coir mat 3 ft. wide. Find the length of matting required supposing the length of the room to be 20 ft.

46. If 5 men can dig a trench 50 ft. long, 10 ft. broad, and 2 yds. deep, in a given time, how many men will be required to dig another trench twice as long, thrice as broad, and 3 yds. deep, in the same time?

47. If 35 men can reap a field in 6 days, in how many days will 42 men be able to reap the same?

48. If a certain quantity of rice is sufficient to feed 45 men for 16 days, how many men can be fed with it for 12 days?

49. If 5 men or 7 boys can reap a field in a certain time, in how many days will 8 boys be able to reap another field which 20 men take 4 days to reap?

50. If 6 men or 9 boys can finish a certain piece of work in a given time, how many men must be associated with 12 boys to finish in half the time a work, twice as great.

51. A clock is set right at 8 o'clock P. M. on Monday, and at 1 o'clock P. M. on Wednesday, it is found to be 3′ too fast.

Supposing its rate regular, what will be the true time when the clock strikes one on Sunday after-noon; and what will be the time by the clock at 1 o'clock P. M. of the same day?

52. Two watches, of which one gains 2′ and the other loses 30′ a day, are both set right at 1 o'clock P. M. on Monday. What will be the difference between the times indicated by them at 9 o'clock A. M. on Sunday, and when will that difference amount to half an hour?

53. If 4 yds. of silk cost as much as 9 yds. of linen, how many yards of linen can be given in exchange for 18 yds. of silk?

54. If 10 masons can build a wall 25 ft. long, 3 ft. high, and 2 ft. thick, in 1 day, in how many days will 15 masons be able to build another wall twice as long, thrice as high, and 3 ft. thick?

55. If 9 men can do a piece of work in 18 days, working 8 hours a day, how many men will be required to finish in 12 days of 6 working hours each, a piece of work 3 times as great?

56. If the carriage of 20 mds. over 5 miles cost 1R. 4a., how many maunds can be carried over 50 miles for 6 Rs. 4a.?

57. If 10 men in 7 days consume 1 md. 30 seers. of rice, what quantity of rice will be required to feed a company of 50 men for the year 1878? And how much more rice must be added to the stock if 12 more men join the company on the 8th of October?

58. If 3 men or 5 boys earn 6 Rs. 9a. in 7 days, how much will 4 men and 6 boys earn in the year 1879?

59. If 18 mds. of gram be sufficient to feed 4 horses for 30 days, what quantity of gram will be required to feed 7 horses for the year 1878?

60. If 24 mangoes can be had for 3Rs., how many mangosteens can be had for 5Rs., supposings 2 mangoes to be worth as much as 5 mangosteens?

61. If 500 men could be fed for 125Rs., 30 years ago, what would the feeding of 600 men cost now, supposing the prices of articles of food to have risen three-fold?

62. A town which is besieged and is defended by 1400 men with provisions enough to sustain them 42 days, supposing each man to receive 18 oz. a day, obtains an increase of 200 men to its garrison on the morning of the 11th day of the siege. What must now be the allowance to each man, in order that the remaining provisions may serve the whole garrison for a further period of 42 days?

63. If 10 cannon which fire 3 rounds in 5 minutes, kill 270 men in $1\frac{1}{2}$ hours, how many cannon which fire 5 rounds in 6 minutes, will kill 500 men in 1 hour at the same rate?

64. A hare starts 40 yards before a greyhound, and is not perceived by him till she has been up 40 seconds: she gets away at the rate of 10 miles an hour, and the dog pursues her at the rate of 18 miles an hour: how long will the course last, and what distance will the hare have run?

65. If a tradesman with a capital of 1800Rs. gain 252Rs. in 7 months, how long will it take him with a capital of 5000Rs. to gain 500Rs.?

66. If a tradesman with a capital of £2700 gain £216 in 6 months, what must be his capital in order that he may gain £1200 in 9 months?

67. At what time between one and two o'clock do the hour and minute hands of a watch point in directions exactly opposite?

68 At what time between twelve and two o'clock are the hands of a clock together again?

69. If a sixpenny loaf weigh 4·35 lbs. when wheat is at 5·75*s.* a bushel, what must be paid for 49·3 lbs. of bread when wheat is at 18·4*s.* a bushel?

70. If 15 men, 12 women, and 9 boys can do a piece of work in 11 days, in what time will 9 men, 12 women and 15 boys be able to finish a piece of work 3 times as great, supposing the parts done by each in the same time to be as the numbers 3, 2, and 1?

CHAPTER VIII.

DIVISION INTO PROPORTIONAL PARTS.

PERCENTAGE, PROFIT AND LOSS, AND AVERAGE. FELLOWSHIP.

SECTION I. DIVISION INTO PROPORTIONAL PARTS.

225. PROP. If there be any number of equal simple fractions, the sum of their numerators divided by the sum of their denominators will give another simple fraction equal to any one of them.

Thus take the simple fractions,

$$\tfrac{2}{3},\ \tfrac{4}{6},\ \tfrac{6}{9}.$$

Then $\because\ \frac{2}{3} = \frac{4}{6} = \frac{6}{9}$,

$\therefore\ 2 = 3 \times \frac{6}{9}$,

and $4 = 6 \times \frac{6}{9}$,

and $\therefore\ 2 + 4 = (3 + 6) \times \frac{6}{9}$.

and $\therefore$ dividing both sides by $3 + 6$,

$$\frac{2+4}{3+6} = \frac{6}{9}.$$

Again $\because\ \dfrac{2+4}{3+6} = \dfrac{6}{9} = \dfrac{2}{3}$,

$\therefore$ in the same way as above

$$\frac{2+4+6}{3+6+9} = \frac{2}{3}.$$

Similarly the proposition may be proved in any other case.

226. Since $\dfrac{2+4+6}{3+6+9} = \dfrac{12}{3+6+9} = \dfrac{2}{3} = \dfrac{4}{6} = \dfrac{6}{9}$,

$\therefore\ 2 : 3 :: 4 : 6 :: 6 : 9$,

and $2 = 3 \times \dfrac{12}{3+6+9}$, $4 = 6 \times \dfrac{12}{3+6+9}$, $6 = 9 \times \dfrac{12}{3+6+9}$.

Thus we see that if we divide any number 12 into parts 2, 4, 6 which are proportional to the numbers 3, 6, 9, these parts are equal to $3 \times \dfrac{12}{3 \times 6 \times 9}$, $6 \times \dfrac{12}{3+6+9}$, and $9 \times \dfrac{12}{3+6+9}$ respectively.

Hence we can deduce the following general Rule :—

RULE. To divide a given number into parts proportional to certain other given numbers, divide the number to be divided by the sum of these latter, and multiply the quotient by each of them ; and the product will be the required part corresponding to that number.

Ex. 1. Divide 27 into parts proportional to 2, 3, 4.

By the Rule,

$\because 2+3+4=9$,

the parts are $2 \times \frac{27}{9} = 6$,

$3 \times \frac{27}{9} = 9$,

and $4 \times \frac{27}{9} = 12$.

Ex. 2. Divide 4500Rs. among A, B, and C so that their shares may be as 3, 5 and 7 respectively.

By the Rule,

$\because 3+5+7=15$,

the share of A $= 3 \times \frac{4500}{15}$ Rs. $= 900$Rs. ;

...............B $= 5 \times \frac{4500}{15}$ Rs. $= 1500$Rs. ;

and..............C $= 7 \times \frac{4500}{15}$ Rs. $= 2100$Rs.

Ex. XLI.

1. Divide

(1) 18 into 3 parts proportional to 1, 2, and 3.
(2) 27 into 3 parts......................2, 3, and 4.
(3) 36 into 3 parts ....3, 4, and 5.
(4) 128 into 4 parts.....................1, 3, 5, and 7.
(5) 1000 into 4 parts....................2, 4, 6, and 8.
(6) 585 into 3 parts.....................11, 13, and 15.

2. Divide 2700 Rs. among A, B, and C so that their shares may be as the numbers 1, 3, and 5.

3. Divide £2400 among A, B, and C, so that as often as A gets £5, B shall get £4, and as often as B gets £8, C shall get £6.

SECTION II. PERCENTAGE, PROFIT AND LOSS, AND AVERAGE.

227. The term *per cent.* means for every hundred.

Thus, if a dealer with a capital of 20Rs. makes a profit of 1R. he makes a profit at the rate of 5Rs. for every 5×20 Rs., or 100 Rs., and he is said to make a profit of 5 *per cent.* on his outlay.

Tradesmen generally estimate their profit and loss by percentages of their capital.

We shall here define some allowances that are made at certain rates per cent. on certain other amounts.

DEFS. COMMISSION is an allowance made to an agent or factor for buying or selling goods for his employer.

BROKERAGE is an allowance made to a broker for effecting the sale of Government Promissory Notes, shares and the like.

PREMIUM is an allowance on the value of goods liable to risk, made to certain parties called *insurers*, who in consideration thereof, undertake in case of loss to make good to the owner the value of the goods insured.

Examples of Profit and Loss and other Examples involving the term per cent., are worked out by the application of the Rule of Three, as will be seen below.

Ex. 1. A grocer buys sugar at 12Rs. a maund, and sells it at 13Rs. What does he gain per cent. on his outlay?

Let x = the gain per cent. required.

Then $\because$ for 12Rs. he gains $(13 - 12)$ R. or 1R.

$$\therefore x : 100 :: 1 : 12$$

$$\text{whence } x = \frac{100}{12} = 8\frac{1}{3}.$$

Ex. 2. How much per cent. is 6 of 15?

Let x = the no. reqd.

Then $\because$ x must bear the same ratio to 100 that 6 bears to 15,

$$\therefore x : 100 :: 6 : 15,$$

$$\text{whence } x = \frac{600}{15} = 40.$$

Ex. 3. If a dealer gains 20 per cent. on his outlay by selling rice at 3Rs. a maund, what was the cost price?

Let x = the cost price reqd. in rupees.

Then $\because$ for every 100Rs. of cost price, the sale brings 120Rs.,

$$\therefore\ x : 3 :: 100 : 120,$$

$$\text{whence } x = \frac{100 \times 3}{120} = \frac{10}{4} = \frac{5}{2};$$

$\therefore$ the cost price = 2Rs. 8a.

Ex. 4. A factor realizes 75Rs. as commission on the selling price of grain at 2 per cent. What was that price?

Let x = the price reqd. in rupees.

Then $x : 100 :: 75 : 2$,

$$\text{whence } x = \frac{75 \times 100}{2} = 3750;$$

$\therefore$ the price reqd. = 3750 Rs.

Ex. 5. Goods worth £ 490 are to be insured at the rate of 2 per cent. To what amount must they be insured so that in case of loss, the value of the goods and the premium paid may be recovered? and what will be the cost of insurance?

If the goods be insured for £ 490 only, then in case of loss the premium paid will be lost, as the party insured will get £ 490 only. If however every £ 100 − £ 2 or £ 98 of the value of the goods be insured for £ 100, then paying £ 2 as premium for the £ 100, the party insured will in case of loss recover £ 100, *i. e.*, £98 (the value of the goods) + £ 2 (the premium paid). Hence putting

x = the reqd. amount in pounds,

we have

$$98 : 100 :: 490 : x,$$

$$\text{whence } x = \frac{100 \times 490}{98} = 500;$$

$\therefore$ the amount reqd. = £ 500.

And the cost of insurance, being 2 per cent. on the amount insured, *i. e.*, $\frac{2}{100}$ of £1 per £1,

$$= \frac{2}{100} \times £\,500 = £\,10.$$

228. DEF. The AVERAGE of several quantities is a quantity which being taken as often as there are quantities will give a sum equal to the sum of the given quantities.

It is therefore found by dividing the sum of the given quantities by their number.

Examples involving the term average may be worked out in the manner given below.

Ex. 1. A tradesman in 3 months gains 625Rs., 590Rs. and 1020Rs. What is his average monthly gain for these 3 months?

$$\text{The reqd. average} = \frac{625 + 590 + 1020}{3} \text{ Rs.}$$

$$= \frac{2235}{3} \text{ Rs.} = 745 \text{ Rs.}$$

Ex. 2. In a class of 9 boys, there are 2 boys each 9 years old, 3 boys each 10 years old, and 4 boys each 11 years old. What is the average age of the boys in the class?

$$\text{The average reqd.} = \frac{2 \times 9 + 3 \times 10 + 4 \times 11}{9} \text{years}$$

$$= \frac{18 + 30 + 44}{9} \text{years}$$

$$= \frac{92}{9} \text{ years}$$

$$= 10\tfrac{2}{9} \text{ years.}$$

Ex. XLII.

1. If the annual value of a holding be 240Rs., and the tax imposed upon it be 18Rs. per annum, at what rate per cent. on the annual value is the tax levied?

2. A dealer buys gram at 1R. 14a. a maund, and sells it at 2Rs. per maund. What does he gain per cent. on his outlay? And at what price per maund must he sell it to secure a profit of 20 per cent.?

3. A dealer by selling goods at 3Rs. 11a. per maund makes a profit of 18 per cent. on his outlay. What was the cost price?

4. A factor realizes 120Rs. as commission on the selling price of grain, at $1\frac{1}{2}$ per cent. What was that price?

5. Goods worth £3900 are to be insured at £$2\frac{1}{2}$ per cent. To what amount must they be insured so that in case of loss the value of the goods and the premium paid may be recovered?

6. A zemindar raises the rent of every ryot by 1 pice in the rupee. By how much per cent. is the income of the zemindari thereby increased?

7. In a class composed of 30 boys, there were present, on Monday 29, on Tuesday 26, on Wednesday 27, on Thursday 25, on Friday 28, and on Saturday 21. What was the average daily attendance during the week?

8. A gentleman earned 9500Rs. from the first of January to the 30th of April; 10800Rs. from the 1st of May to the 30th of September; and 4300Rs. from the 1st of October to the 31st of December. What was his average monthly income during the year?

SECTION III. FELLOWSHIP.

229. Defs. Fellowship, called also Partnership, is a method by which the profits or losses of partners in any trade or business are determined.

It is called Simple Fellowship or Compound Fellowship according as the capitals of the several partners remain invested in the joint trade for the same period or for different periods of time.

Examples of Fellowship are only particular instances of Division into Proportional Parts, and are worked out in the manner given below.

1. Simple Fellowship.

Ex. Two partners A and B contribute 5000Rs. and 6000Rs. respectively for there joint business, and make a profit of 990Rs. What share of the profit will each get?

As the profit is to be divided in proportion to the capital contributed by each, the question is reduced to dividing 990 into parts proportional to 5000 and 6000.

Hence the share of $A = 5000 \times \dfrac{990}{5000 + 6000}$ Rs.

$= 450$ Rs.

and.................$B = 6000 \times \dfrac{990}{11000}$ Rs.

$= 540$ Rs.

II. Compound Fellowship.

Ex. In a joint trade, A contributes the sum of 4000Rs. which remains, invested for 5 months, and B contributes 3000 Rs. which remains invested for 6 months. They make a profit of 570 Rs. What is the share of each?

The sum of 4000 Rs. invested for 5 months is the same as 5×4000 Rs.................1 month;
and similarly 3000 Rs.................6 months is the same as 6×3000 Rs.................1 month
and thus the question is reduced to one of Simple Fellowship where the capitals of the partners are 5×4000Rs. and 6×3000Rs., *i. e.*, 20000Rs. and 18000Rs.

Consequently, the share of $A = 20000 \times \dfrac{570}{38000}$ Rs.

$= 300$ Rs.

and ....$B = 18000 \times \dfrac{570}{38000}$ Rs.

$= 270$Rs.

Ex. XLIII.

1. Two partners in trade contribute respectively 4000Rs. and 5000Rs., and they gain 1350Rs. How ought the profit to be divided between them?

2. A trading firm with a capital of £25000, is composed of three partners, A, B, and C. A owns £6000, B, £9000, and C, the remainder of the capital. If the profits of the firm amount to £3000, how much of it will each get?

3. A trading firm is composed of two partners A, and B. For every rupee that A owns in the capital, B owns three rupees. How ought a profit of 2400Rs. to be divided between them?

4. *A* opens a shop with a capital of 2000Rs. Four months after, *B* joins with a capital of 3000Rs., and two months after *B*'s joining, *C* adds a capital of 4500Rs. The profits of the shop at the end of a year, amount to 900Rs. How ought the amount to be divided?

5. In the preceding Example, if *A* puts in a farther sum of 500Rs. when *C* joins, and the profits amount to 910Rs., how much of the amount will each get?

6. *A* buys an estate yielding an income of 12000Rs. per annum, and 4 months after his purchase, sells a five annas share of it to *B*. How ought the income to be divided at the end of the year?

CHAPTER IX.

INTEREST.

230. DEFS. INTEREST is money paid for the use of money.

The sum lent is called the PRINCIPAL.

The principal together with the interest for any period is called the AMOUNT.

Interest is generally reckoned at a certain *rate* per cent. *per annum*, *i. e.*, for a year. In this country interest is often reckoned at a certain rate per cent. per *mensem*, *i. e.*, for a month.

When interest is charged on the principal alone, it is called SIMPLE INTEREST.

When interest remains unpaid and is added to the principal as soon as it is due, and then interest is charged on the whole, it is called COMPOUND INTEREST.

231. When interest has to be calculated from one given day to another, as for instance from the 15th of August to the 2nd of December, the first day *i. e.*, the 15th of August is left out, but the last day *i. e.*, the 2nd of December is taken into account. This is the rule given in English books on Arithmetic. But in this country the practice is just the reverse, interest being charged for the day of borrowing and not for the day of repayment. The result, however, would be the same in both cases.

232. Since Amount = Principal + Interest,
$\therefore$ Interest = Amount − Principal,
and Principal = Amount − Interest.

SECTION I. SIMPLE INTEREST.

233. *Given the principal, the rate, and the time, to find the interest.*

Ex. 1. Find the interest on £325 for 3 years at 6 per cent. per annum.

Let x = the interest reqd. in pounds.

Then $\because$ £100 give £6 in one year.

$\therefore$ £100 will give £ (3×6) in 3 years;

and as £325............£ x in the same period of 3 years, at the same rate,

$\therefore$ 100: 325 :: $3 \times 6 : x$;

and $\therefore x = \dfrac{325 \times 3 \times 6}{100} = \dfrac{117}{2} = 58\frac{1}{2}$;

$\therefore$ the interest reqd. = £58. 10*s.*

Ex. 2. The sum of 650Rs. is lent on the 2nd of December 1872. Find the interest due on the 16th of August 1875, at 5 per cent. per annum.

Here the time for which interest is due is the time from 2nd December 1872 to 16th August 1875, and in reckoning that time, the 2nd of December is to be included and the 16th of August is to be excluded. Now the time from 2nd December 1872 to 1st December 1874 is 2 years, and that from 2nd December 1874 to 15th August 1875, both days inclusive, is

$(30 + 31 + 28 + 31 + 30 + 31 + 30 + 31 + 15)$ days or 257 days;

$\therefore$ the whole time is 2 years and 257 days or $2\frac{257}{365}$ years. Hence as in Ex. 1,

the interest reqd. $= \dfrac{650 \times 2\frac{257}{365} \times 5}{100}$ Rs.

In practice it is usual to calculate the interest for the year *and that for the days separately.*

Thus in the above Example,

the interest for 2 years $= \dfrac{650 \times 2 \times 5}{100}$ Rs. = 65Rs.

and..............257 days $= \dfrac{650 \times \frac{257}{365} \times 5}{100}$ Rs. $= \dfrac{650 \times 257 \times 5}{100 \times 365}$ Rs.

$= \dfrac{13 \times 257}{2 \times 73}$ Rs. $= \dfrac{3341}{146}$ Rs.

$= 22\dfrac{129}{146}$ Rs.;

$\therefore$ the whole interest $= 87\dfrac{129}{146}$ Rs.;

Ex. 3. Find the interest on 750 Rs. 8 a. for 2 years and 10 months at $7\frac{1}{2}$ per cent. per annum.

In Examples like these, where the months are not named, 1 month is taken to be $\frac{1}{12}$ of a year.

Hence in this case, the no. of years $= 2\frac{10}{12} = 2\frac{5}{6}$;
and the principal $= 750$ Rs. 8 a. $= 750\frac{1}{2}$ Rs.

Therefore as in Ex. 1,

$$\text{the interest reqd.} = \frac{750\frac{1}{2} \times 2\frac{5}{6} \times 7\frac{1}{2}}{100} \text{ Rs.}$$

$$= \frac{1501 \times 17 \times 15}{2 \times 6 \times 2 \times 100} \text{ Rs.} = \frac{25517}{160} \text{ Rs.}$$

$$= 159 \text{ Rs. } 7\text{a. } 8\tfrac{2}{5}\text{p.}$$

From the above we deduce the following Rules:

RULE I. To calculate the interest for any number of years integral or fractional, multiply the principal by the number of years, and the product by the rate per cent. per annum, and divide the result by 100; the quotient will be the interest expressed in the same denomination as the principal.

RULE II. To calculate the interest for any number of days, multiply the principal by the number of days, and the product by the rate per cent. per annum and divide the result by 100×365; and the quotient will be the interest required.

234. From the preceding Article we see that if the rate be a rate per cent. per annum, and the time be expressed in years,

$$\text{Interest} = \frac{\text{Principal} \times \text{Time} \times \text{Rate}}{100} \quad \text{............(1)}$$

$\therefore$ $100 \times \text{Interest} = \text{Principal} \times \text{Time} \times \text{Rate}$,
and $\therefore$ dividing both sides by Principal $\times$ Rate, we have

$$\text{Time} = \frac{100 \times \text{Interest}}{\text{Principal} \times \text{Rate}} \quad \text{......................(2)}$$

Again, dividing both sides by Principal $\times$ Time, we have

$$\text{Rate} = \frac{100 \times \text{Interest}}{\text{Principal} \times \text{Time}} \quad \text{......................(3)}$$

Lastly, dividing both sides by Time $\times$ Rate, we have

$$\text{Principal} = \frac{100 \times \text{Interest}}{\text{Time} \times \text{Rate}} \quad \text{.........................(4)}$$

An equation like any of the preceding, (1), (2), (3), or (4), is called a *Formula*.

From the four formulæ given above, we see that when any three of the four quantities, principal, interest, rate, and time, are given, the fourth can be found.

235. We can obtain the above formulæ independently of Art 233. Thus take formula (2).

Let it be required to find in how many years £550 will amount to £616 at 6 per cent. per annum.

Let x = the no. of years reqd.

Then in x years £100 will give $x \times$ £6 as interest
and £550............£616 − £550 or £66
at the same rate,

$\therefore 100 : 550 :: x \times 6 : 66,$

and $\therefore x \times 6 \times 550 = 100 \times 66,$

$$\text{whence } x = \frac{100 \times 66}{550 \times 6},$$

$$i.\ e., \text{ the time reqd.} = \frac{100 \times \text{interest}}{\text{principal} \times \text{rate}}.$$

Similarly by the application of the principle of proportion, the other two formulæ (3) and (4) can be obtained independently of Art. 233.

236. *Given the amount, the rate and the time, to find the principal.*

Let it be required to find what sum will amount to 616 Rs. in 2 years at 6 per cent. per annum.

Let x = the sum reqd.

Then $\because$ 100 Rs. in 2 years amount to $(100 + 2 \times 6)$ Rs.
and x Rs...............................616 Rs.
at the same rate of interest, *i. e.* at the same rate of increase,

$\therefore 100 : x :: 100 + 2 \times 6 : 616,$

$$\text{whence } x = \frac{616 \times 100}{100 + 2 \times 6}$$

$$i.\ e.\ \text{Principal} = \frac{\text{Amount} \times 100}{100 + \text{Time} \times \text{Rate}}$$

237. When partial payments are made on different dates, the interest for each interval is calculated on the portion of the principal that remains due during it, as will be seen from the Example given below.

Ex. The sum of 600 Rs. is borrowed on the 3rd of January 1874; and on the 7th of February and the 21st of April following, payments of Rs. 50 and Rs. 100 respectively are made in liquidation of the principal. Find the amount due on the 21st of May following, at 5 per cent. per annum simple interest.

The process may be stated thus :—

From 3rd Jan. to 6th Feb. both days inclusive *i. e.*, for 35 days	Rs. Principal $=600$	Rs. Int. $=2\frac{64}{73}$
From 7th Feb. to 20th Apr. both days inclusive *i. e.*, for 73 days	.... $=600-50$ $=550$	.... $=5\frac{1}{2}$
From 21st Apr. to 20th May both days inclusive. *i. e.*, for 30 days	.... $=550-100$ $=450$	... $=1\frac{67}{73}$
		Total $10\frac{33}{146}$

Thus the reqd. amount due $=450$Rs. $+\ 10\frac{33}{146}$Rs. $=\ 460\frac{33}{146}$Rs.

We have supposed the payments to be made in liquidation of the principal only. If however the creditor has the choice of appropriating the payments, and he takes them in liquidation of the interest due as well as the principal, the process will stand thus :—

For the 1st period, Principal $=600$ Rs. Int. $=2\frac{64}{73}$ Rs.

............2nd .... $=600$ Rs. $-\ (50-2\frac{64}{73})$ Rs.

$=552\frac{64}{73}$ Rs.; Int. $=5\frac{193}{365}$ Rs.

............3rd .... $=552\frac{64}{73}$ Rs. $-\ (100-5\frac{193}{365})$Rs.

$=458\frac{148}{365}$Rs. Int. $=1\frac{235504}{266450}$Rs.

Here we need not add the interests, as the last interest is the only one that is due, the others having been paid up.

$$\text{Thus, the amount due} = \left(458\frac{148}{365}+1\frac{235504}{266450}\right)\text{Rs.}$$

$$=460\frac{77094}{266450}\ \text{Rs.}$$

238. When there is a mutual account current between two persons, so that each may be regarded as a lender and a borrower in respect of the sums paid and received by him respectively, the payments and receipts by either of them,

(which will be the same as the receipts and payments by the other,) are kept separate, and on opposite sides of the account, and interest runs on each side, for each period on the sum that remains due for that period, as will be seen from the Example given below.

Ex. *A* receives from *B* on Mar. 3, 1874, 10,000 Rs.	*B* receives from *A* on Apr. 2, 1874, 100 Rs.
... May 12, ..., 2,000	... Apr. 22, ... 9100......

Find the balance due to *B* on May 17, allowing interest at $18\frac{1}{4}$ per cent. per annum.

The calculation will stand thus :—

On the left side of the account—

From Mar. 3 to May 11 both days inclusive *i. e.*, for 70 days } Principal = 10000 Rs., Int. = 350 Rs.

From May 12 to May 16 both days inclusive, *i. e.*, for 5 days } Principal = 12000 Rs., Int. = 30 Rs.

Total 380 Rs.

Thus on May 17, the total sum due from *A* to *B*
= (12000 + 380) Rs. = 12380 Rs.

On the right side of the account—

From Apr. 2, to Apr. 21 both days inclusive *i. e.* for 20 days } Principal = 100 Rs., Int. = 1 R.

From Apr. 22 to May 16 both days inclusive *i. e.* for 25 days } = 9200 Rs. = 115 Rs

Total 116 Rs.

Thus on May 17 the total sum due from *B* to *A*
= (9200 + 116) Rs. = 9316 Rs.

Therefore on May 17, the balance due to *B*
= (12380 − 9316)Rs. = 3064 Rs.

The above mode of calculating interest is called in this country the *Ganga-Jamuna* mode, because interest on sums paid by *B*, and that on sums paid by *A*, run side by side like the Ganges and the Jumna.

If we regard *A* alone as the borrower and *B* the lender having the option of appropriating payments in his own way, the account will stand differently thus :—

From Mar. 3 to Apr. 1 both days inclusive *i. e.* for 30 days } Principal = 10,000Rs.; Int. = 150 Rs.

On Apr. 2, after payment of 100 Rs. Int. due = 50 Rs.

From Apr. 2 to Apr. 21 both days inclusive *i. e.*, for 20 days } Principal = 10000Rs.; Int. = 100 Rs.

On Apr. 22 Int. due = 150 Rs.

From Apr. 22 to May 11 both days inclusive *i. e.*, for 20 days } ... = 10000Rs. (9100 – 150) Rs.

= 1050 Rs.; Int. = $10\frac{1}{2}$ Rs.

From May 12 to May 16 both days inclusive *i. e.*, for 5 days } ... = 1050 Rs. + 2000 Rs.

= 3050 Rs.; Int. = $7\frac{5}{8}$ Rs.

Therefore on May 17, the amount due to *B*

$$= (3050 + 10\tfrac{1}{2} + 7\tfrac{5}{8}) \text{ Rs.}$$
$$= 3068\tfrac{1}{8} \text{ Rs.}$$

Hence it will be seen that the Ganga-Jamuna mode is more advantageous to the borrower than the latter mode; and the reason is obvious. For whereas, in the former mode, the sum of Rs. 100 paid by *A* on Apr. 2. carries interest in his favor, in the latter mode, being taken in liquidation of interest due, it does not carry any interest in his favor. For convenience of calculation, the former mode is often adopted instead of the latter.

Ex. XLIV.

1. Find the simple interest and the amount

(1) Of 75Rs. for 1 year at 6 per cent. per annum.
(2) Of 80Rs. for 2 years at 9..........................
(3) Of 125Rs. for $2\frac{1}{2}$ years at $7\frac{1}{2}$..........................
(4) Of 2560Rs. for 4 years at 12..........................
(5) Of £1050 10s. for 5 years at 4..........................
(6) Of £5500 for $3\frac{1}{2}$ years at $4\frac{1}{2}$..........................
(7) Of 1050Rs. for $2\frac{1}{4}$ years at $1\frac{1}{2}$ per cent. per mensem.
(8) Of 750Rs. for 2 years at $1\frac{1}{4}$..........................

2. Find the simple interest

(1) On 60 Rs. for 4 months at 2 per cent. per mensem.
(2) On 85 Rs. for 9 months at $1\frac{9}{16}$........................
(3) On 225 Rs. for 10 months at 10 per cent. per annum
(4) On 520 Rs. for 8 months at 9.........................
(5) On 120Rs. for 7months at 1pice per rupee per month
(6) On 56 Rs. for 6 months at 2..... ....
(7) On 1200 Rs. from October 18, 1878 to March 21, 1879, at 6 per cent. per annum.
(8) On 1560 Rs. from July 21, 1870 to June 20, 1871 at 9 per cent. per annum.
(9) On £500 from March 20 to August 31, at 5 per cent. per annum.

3. In what time will

(1) £1000 amount to £1500 at 5 per cent. per annum?
(2) £625 amount to £800 at 4........................ ?
(3) 120Rs. amount to 200Rs. at 10...................... ?
(4) 40000 Rs. amount to 50000 Rs. at 4.................. ?
(5) 80 Rs. amount to 100 Rs. at $1\frac{9}{16}$ per cent. per mensem?
(6) 125 Rs. amount to 200 Rs. at $1\frac{1}{4}$ per cent. per mensem?

4. At what rate of simple interest will

(1) 60 Rs. amount to 80 Rs. in 2 years?
(2) 75 Rs................100 Rs. in $2\frac{1}{2}$ years?
(3) £4000...............£5000 in 8 years?
(4) £640...............£700 in 5 years?
(5) 1000Rs.............1250Rs. in 4 years?
(6) 2225 Rs.............2670 Rs. in 2 years?

5. Find the sum of which the interest is

(1) 60 Rs. in 3 years at 4 per cent. per annum.
(2) £80 in 2 years at 5..........................
(3) £100 in 5 years at 4..........................
(4) 600Rs. in 2 years at 4..........................
(5) 10000Rs. in 5 years at 6......................
(6) 1664Rs. in 8 years at 13......................

6. What sum will amount to

(1) £100 in 3 years at 4 per cent. per annum ?
(2) £625 in 5 years at 5 .... ?
(3) 10000Rs. in 4 years at 12........................ ?
(4) 25000Rs. in 6 years at 10......................... ?

7. In what time will a sum double itself at 4 per cent. per annum ?

8. At what rate will a sum double itself in 5 years ?

SECTION II. COMPOUND INTEREST.

239. Rule. To find the compound interest of a given sum for a given time at a given rate, find the interest of the given sum for the first year, and add it to the principal for that year; the sum will be the principal for the second year. Find the interest of this for the second year, and add it to the principal for that year; the sum will be the principal for the third year. Proceed in this way, and the sum of the interests for the several years will be the compound interest required.

The reason for this Rule is obvious from the definition of compound interest.

Ex. Find the compound interest and the amount of 325Rs. for 3 years at 5 per cent. per annum.

We have,

principal for 1st year $= 325$ Rs.,

int.................. $= \dfrac{325 \times 5}{100} = \underline{16.25}$ Rs.;

principal......2nd...... $= 341.25$Rs.,

int.................. $= \dfrac{341.25 \times 5}{100} = \underline{17.0625}$ Rs. ;

principal for 3rd year $= 358.3125$Rs.

int.................. $= \dfrac{358.3125 \times 5}{100} = \underline{17.915625}$ Rs.

$\therefore$ amount at the end of the 3rd year $= 376.228125$Rs.,

and the int. reqd. $= (16.25 + 17.0625 + 17.915625)$ Rs.

$= 51.228125$Rs.

$= 51$Rs. 3a. $2\frac{3}{5}$ pice.

To avoid fractions, the calculation is usually made in decimals.

240. It is customary, when compound interest for a number of entire years and for a part of a year, as for instance for $2\frac{3}{4}$ years, is required, to find the interest for the last or the 3rd year, and then to take $\frac{3}{4}$ of it as the interest for the $\frac{3}{4}$ths of the 3rd year.

When interest is payable oftener than once a year, the interest due at the end of each interval will have to be added to the principal for that interval, as will be seen from the Example given below.

Ex. Find the compound interest of £250 for 2 years at 4 per cent. per annum, interest being payable half-yearly.

We have

principal for 1st half year		= £250.
int..........................	$=\frac{250 \times 4}{2 \times 100}$	= £ 5·00
principal....2nd............		= £255.
int..........................	$=\frac{255 \times 4}{2 \times 100}$	= £ 5·1
principal....3rd...........		= £260·1
int..........................	$=\frac{250 \cdot 1 \times 4}{2 \times 100}$	= £ 5·202
principal....4th...........		= £265·302
int..........................	$=\frac{265 \cdot 302 \times 4}{2 \times 100}$	= £ 5·30604

∴ the compound interest reqd. = £(5 + 5·1 + 5·202 + 5·30604)
= £20·60804.

Ex. XLV.

Find the compound interest and the amount

1. Of 80Rs. for 2 years at 10 per cent. per annum.
2. Of 75Rs. for $2\frac{1}{2}$ years at 8..............................
3. Of £125 for 2 years at 4..............................
4. Of £500 for 3 years at 10..............................
5. Of 2000Rs. for 2 years at 4..............................
6. Of 25000Rs. for 3 years at 10..............................

CHAPTER X.

PRESENT WORTH AND DISCOUNT.

241. When a sum of money is payable at the end of a given time, and the debtor, instead of waiting for that time, wishes to pay off the debt immediately, it is clear that he ought to pay, not the whole amount due, but something less, as he will not have the use of the money for the given time, and the creditor, who will have that use, may, by investing the money at the current rate of interest, get at the end of the given time an amount equal to what was then payable originally.

DEFS. The amount which the creditor is entitled to receive when payment is made before it is due, is called the PRESENT VALUE or the PRESENT WORTH of the amount due.

The allowance made on any sum payable after some time, when it is paid before it becomes due, is called DISCOUNT.

Hence,

Present Worth = Given sum − Discount.

and Discount = Given sum − Present Worth

= Int. on Present Worth for the given time at the assumed rate.

242. *To find the present value of a given sum due at the end of a given time, at a given rate of simple interest.*

Let it be required to find the present value of £750 due 3 years hence, at 5 per cent. per annum simple interest.

Let x = the present value reqd. in pounds.

Then ∵ £100 in 3 years at 5 per cent. will amount to £100 + £(3 × 5)

∴ £100 is the present value of £ (100 + 3 × 5) due 3 years hence at 4 per cent.

and £ x is the present value of £ 750 due 3 years hence at the same rate;

$$\therefore\ 100 : x :: 100 + 3 \times 5 : 750.$$

$$\text{whence } x = \frac{100 \times 750}{100 + 3 \times 5},$$

$$\text{i. e., Present Value} = \frac{100 \times \text{Given sum.}}{100 + \text{Time} \times \text{Rate}}.$$

Ex. A tenant has to pay a rent of 240Rs. at the end of the year. If he pays at the beginning of the year, what sum will suffice, supposing the rate of simple interest to be 12 per cent. per annum?

By the formula,

$$\text{the amount reqd.} = \frac{100 \times 240}{100 + 1 \times 12} \text{ Rs.}$$

$$= \frac{100 \times 240}{112} \text{ Rs.}$$

$$= \frac{1500}{7} \text{ Rs.}$$

$$= 214 \text{ Rs. } 4 \text{ a. } 2\tfrac{2}{7} \text{ pice.}$$

243. *To find the discount on a given sum due at the end of a given time at a given rate of simple interest.*

Since discount = given sum − present value,

∴ the discount is found by first finding the present worth (Art. 242) and then subtracting it from the given sum.

It may also be found independently thus :

Let it be required to find the discount on 250 Rs. due 2 years hence, at 5 per cent. per annum simple interest.

Let x = the discount reqd.

Then ∵ £100 in 3 years will amount to £$(100 + 3 \times 5)$

∴ £(3×5) is the discount on £$(100 + 3 \times 5)$ due 3 years hence at 5 per cent.

and £ x..............................£250.....................;

$$\therefore \quad x : 3 \times 5 :: 250 : 100 + 3 \times 5$$

$$\text{and } \therefore x = \frac{250 \times 3 \times 5}{100 + 3 \times 5},$$

$$\text{i. e., Discount} = \frac{\text{Given sum} \times \text{Time} \times \text{Rate}}{100 + \text{Time} \times \text{Rate}}.$$

Ex. If the credit price of a set of books to be paid after 6 months be 150 Rs ; what deduction will be made when the price is paid in cash, supposing the rate of simple interest to be 18 per cent. per annum?

By the formula,

the deduction or discount reqd. $= \dfrac{150 \times \frac{1}{2} \times 18}{100 + \frac{1}{2} \times 18}$ Rs.

$= \dfrac{150 \times 9}{109}$ Rs.

$= \dfrac{1350}{109}$ Rs.

$= 12$ Rs. 6a. $1\dfrac{107}{109}$p.

244. Since the present value of any sum is less than that sum,

and since the discount on any sum is the interest on its present value,

$\therefore$ the discount on any sum payable after any given time is less than the interest on that sum for that time at the given rate.

The distinction between discount and interest will be made clear by an Example.

Ex. If 5 copies of a certain book can be had for a certain sum payable at the end of a year, and 6 copies for the same sum paid immediately, find the rates of discount and interest.

Since cash price of 6 copies = credit price of 5 copies,

$\therefore$ 6 × cash price of 1 copy = 5 × credit price of 1 copy,

and $\therefore$ cash price of 1 copy $= \frac{5}{6}$ credit price of 1 copy

= credit price of 1 copy

$- \frac{1}{6}$ of 1 credit price of 1 copy.

Hence in 1 year, discount on credit price $= \frac{1}{6}$ of that price

and $\therefore$ .... 100 Rs. $= \frac{1}{6}$ of 100 Rs. $= 16\frac{2}{3}$ Rs.

or $16\frac{2}{3}$ per cent. is the rate of discount.

Again,

$\because$ 5 × credit price of 1 copy = 6 × cash price of 1 copy

$\therefore$ credit price of 1 copy $= \frac{6}{5}$ of cash price of 1 copy.

= cash price of 1 copy

$+ \frac{1}{5}$ of cash price of 1 copy.

Hence in 1 year,

interest of the cash price $= \frac{1}{5}$ of that price

and .... 100 Rs. $= \frac{1}{5}$ of 100 Rs. = 20 Rs.

or 20 per cent. per annum is the rate of interest.

245. DEFS. A BILL OF EXCHANGE or a HUNDI is a writing by which one person directs another to pay a certain sum of money to a third at a certain time.

A PROMISSORY NOTE is a writing by which a person promises to pay a certain sum of money at a certain time.

Bills of exchange, promissory notes, and hundis are instances in which money is payable at a future period. In cases of bills of eechange and promissory notes, except those payable on demand, after due date, three additional days called *days of grace* are allowed by the law of England so that a bill or a note becomes *legally* due three days after it becomes *nominally* due. In cases of hundis in this country, merchants and bankers usually allow three days of grace.

A bill or a note drawn on the last day of any month and made payable a certain number of months after date, will become nominally due on the last day of the last month whether it be the 28th, 29th, 30th or 31st, and legally due on the 3rd of the next month. When the day on which a bill with or without the days of grace is due falls on a Sunday, Good Friday, or Christmas day in England, the bill or note becomes due on the previous day.

246. When a banker makes cash payment to the holder of a bill for a given sum payable after a given time, it is customary to deduct the interest on that sum for the given time instead of the discount; and as by Art. 244, the interest is always greater than the discount, the transaction is always advantageous to the banker.

Ex. A bill of £500 is drawn on April 3, 1874 at 6 months date, and is discounted on June 25, at 6 per cent. What does the banker gain by the transaction?

Adding the 3 days of grace the bill falls due on the 6th of September 1874; and from the 25th of Jnne to the 6th of September, there are

$$6 + 31 + 31 + 5 \text{ or } 73 \text{ days};$$

$$\therefore \text{interest deducted} = £\,\frac{500 \times 73 \times 5}{100 \times 365} = £5,$$

$$\text{and the true discount} = £\frac{500 \times \frac{73}{365} \times 5}{100 + \frac{73}{365} \times 5} = £\frac{500}{101} = £4\frac{96}{101};$$

$$\therefore \text{the banker's gain} = £\left(5 - 4\frac{96}{101}\right) = £\,\frac{5}{101} = 11\frac{89}{101}d.$$

Ex. XLVI.

1. Find the present worth of

(1). 100Rs. due 1 year hence at 12 per cent. per ann. simp. int.
(2). £200 due 2 years hence at 5............................
(3). £784 due 3 years hence at 4............................
(4). 1020Rs. due 4 years hence at 9........................
(5). 575Rs. due 2 years hence at $7\frac{1}{2}$........................
(6). 2623Rs. due 12 years hence at 6........................

2. Find the discount on

(1). £260 due 1 year hence at 4 per cent. per ann. simp. int.
(2). £1045 due 2 years hence at 5 ....
(3). 1239Rs. due 3 years hence at 6........................
(4). 1560Rs. due 4 years hence at $7\frac{1}{2}$........................
(5). 1000Rs. due 5 years hence at 10........................
(6). 1250Rs. due 6 years hence at $12\frac{1}{2}$........................

3. A bill of £ 500 drawn on the 13th of March, and payable 6 months after date, is discounted on June 30th at 4 per cent. What does the banker gain by the transaction?

4. *A* grants a lease of his zemindary to *B* for 3 years *B* after paying all expenses and the rent reserved by the lease. has a clear profit of 1200 Rs. a year. If at the end of a year, *B* agrees to give back the estate to *A* upon receipt of proper compensation, what ought the amount of such compensation to be, supposing the ordinary rate of interest to be 6 per cent. per annum?

5. The Prem Chand Roy Chand Studentship is worth 2000 Rs. a year, and is tenable for 5 years. For what sum paid down immediately, ought a successful candidate to commute it, if the rate of interest be 4 per cent. per annum?

6. If the cash price of a book be 5Rs. 5a., and its credit price payable at the end of a year, 6Rs. 4a., find the rates of interest and discount.

CHAPTER XI.

EQUATION OF PAYMENTS.

247. Defs. When several debts are due from one person to another after different periods of time, the time after which all the debts may be paid at once without loss to either party is called the Equated Time of Payment.

To satisfy the condition of fairness to both parties, the equated time ought to be such that the sum of the present values of the several debts due after their respective periods is equal to the present value of the sum of those debts due after the equated time. In practice, however, the condition of fairness is supposed to be satisfied if the sum of the interests on the several debts for their respective times is equal to the interest on the sum of those debts for the equated time. On this supposition the rule for finding the equated time may be deduced from the following Example.

Ex. Three sums 100Rs. 200Rs. and 300Rs. are due after 2, 3 and 4 years respectively. Find the equated time of payment.

Let x = the equated time reqd. in years.

Then by the supposition,

int. on 100Rs. for 2 years + int. on 200Rs. for 3 years + int. on 300Rs. for 4 years.

= int. on (100 + 200 + 300)Rs. for x years;

$$\therefore \frac{100 \times 2 \times \text{Rate}}{100} + \frac{200 \times 3 \times \text{Rate}}{100} + \frac{300 \times 4 \times \text{Rate}}{100}$$
$$= \frac{(100 + 200 + 300) \times x \times \text{Rate}}{100}$$

or multiplying both sides by 100 and dividing by the Rate,

$$100 \times 2 + 200 \times 3 + 300 \times 4 = (100 + 200 + 300) \times x;$$

$$\therefore x = \frac{100 \times 2 + 200 \times 3 + 300 \times 4}{100 + 200 + 300}.$$

Hence we get the following Rule.

Rule. Multiply each debt by the time after which it is due, and divide the sum of these products by the sum of the debts. The quotient will be the equated time required.

Ex. XLVII.

1. *A* owes *B* 100Rs., whereof the sum of 20Rs. is to be paid in 3 months, and the remainder in 4 months. Find the equated time.

2. A debt of £1000 was payable at the end of a year. The debtor however pays £200 after 3 months, and £300 after 4 months. When ought the remainder to be paid?

3. A tenant pays a yearly rent of 500Rs. according to the following instalments:—100Rs. at the end of 3 months; 100Rs. at the end of 6 months; 200Rs. at the end of 7months; and 100Rs. at the end of the year. When ought he to pay the whole rent if he pays it in one sum?

4 *A* owes *B* a debt of 4800Rs., one-half of which is due in 3 months, one-third in 4 months, and the remainder in 8 months. Find the equated time.

5. Find the equated time of payment of £960, $\frac{1}{8}$ of which is due in 7months, $\frac{1}{6}$ in 9 months, and the rest in 18 months.

6. *A* owes *B* a debt payable in 8 months, but he pays $\frac{1}{3}$ of the debt in 4 months, and $\frac{1}{4}$ in 5 months; when ought the remainder to be paid?

CHAPTER XII.

STOCKS.

248. DEF. STOCK means the capital of banks or trading companies; or the capital borrowed by any Government to meet national expenses.

In the latter case, it is called the *Funds* or the *National Debt*, and is also called in England *consols*, a contraction for *consolidated annuities*, and in India, Government Securities or Government Promissory Notes.

When any Government raises capital by borrowing, it generally reserves to itself the option of paying off the principal at any future time, promising however to pay interest regularly at fixed periods. In India, the interest on Government Securities is generally paid half-yearly.

Banks and trading companies make periodical distributions of their profits amongst their shareholders, the portions of the profit given to the shareholders being called *dividends*.

249. Stock is transferable by sale, and at any time can be converted into money. But its price is continually fluctuating, depending upon a variety of causes, and amongst others, upon the amount of money available in the market for investment in Stock.

Stock is said to be at *a premium*, at *par* or at *a discount* according as the price of 100Rs. stock is greater than, equal to, or less than 100Rs. in money.

When 100Rs. stock at 4 per cent. is sold for any sum, such as 105Rs. or 100Rs. or 98Rs. as the case may be, for every 105Rs. or 100Rs. or 98Rs. the purchaser will get 4Rs. per annum as interest from Government.

Stock is bought and sold through brokers who generally charge $\frac{1}{8}$R. on every 100Rs. stock bought or sold. Thus, the purchaser of the 4 per cents at 102 will have to pay 102Rs. + $\frac{1}{8}$R or 102$\frac{1}{8}$Rs. for 100Rs. stock purchased.

In working out Examples however, the brokerage, if not mentioned, need not be taken into consideration.

250. The different classes of Examples in Stocks can be worked out by the aid of the principles of proportion. We shall give one or two instances of each.

I. VALUE OF STOCK.

Ex. 1. What amount of stock in the 4 per cents. at 95 can be bought for 19000 Rs. ?

Let x = amount reqd. in rupees.

Then $\because$ 95Rs. is the price of 100 Rs. stock,

$\therefore 95 : 19000 :: 100 : x$

$$\text{and} \therefore x = \frac{19000 \times 100}{95} = 20000,$$

and the amount of stock reqd. = 20000 Rs.

Ex. 2. When the 5 per cents. are at 102, what is the cost of purchasing £2720 in the 5 per cents., brokerage being $\frac{1}{8}$ per cent.?

Let x = the cost reqd. in pounds.

Then $\because$ £100 stock cost £$102\frac{1}{8}$,

$\therefore 100 : 2720 :: 102\frac{1}{8} : x$

$$\text{and} \therefore x = \frac{2720 \times 817}{8 \times 100} = \frac{17 \times 817}{5} = 2777\tfrac{4}{5},$$

and the price reqd. = £2777. 16*s.*

II. INTEREST ON STOCK.

Ex. 3. What is the interest on Government Securities for 19500Rs. at $4\frac{1}{2}$ per cent. for 2 years ?

$$\text{The int. reqd.} = \frac{19500 \times 4\frac{1}{2} \times 2}{100}\text{Rs.} = 1755\text{ Rs.}$$

Ex. 4. What annual income can be secured by investing 17000Rs. in the 4 per cents. at 102 ?

Let x = income reqd. in rupees.

Then $\because$ 102Rs. give 4 Rs.,

$\therefore 102 : 17000 :: 4 : x$;

$$\text{whence } x = \frac{17000 \times 4}{102} = 666\tfrac{2}{3},$$

and $\therefore$ income reqd. = $666\frac{2}{3}$ Rs.

III. COMPARISON AND TRANSFER OF STOCK.

Ex. 5. If the 4 per cents. be at 92 and the 5 per cents. at 105, which is the more profitable investment of the two ?

In the former case 92 Rs. give 4 Rs.

or 1 R. gives $\frac{4}{92}$ R.

In the latter case 105Rs. give 5Rs.

or 1 R. gives $\frac{5}{105}$ R.

And as $\frac{5}{105}$ is greater than $\frac{4}{92}$,

$\therefore$ the latter is the more profitable investment of the two.

Ex. 6. What change in income is produced by the transfer of 7500 Rs. stock from the 4 per cents. at 98 to the 6 per cents. at 105 ?

The income before transfer $= \dfrac{7500 \times 4}{100} = 300$ Rs.

Now let $x =$ the amount in rupees of the 6 per cents. that can be bought for 7500 Rs. in the 4 per cents.

Then $\because$ every 100 Rs. stock in the 6 per cents. costs 105 Rs.
and....................................4.......................98...
and the quantity of stock that can be bought for a given amount varies inversely as the price per 100 Rs. stock.

$\therefore$ 98 : 105 :: x : 7500 ;

whence $x = \dfrac{7500 \times 98}{105} = 7000$;

$\therefore$ the transfer gives 7000 Rs. in the 6 per cents.

and the income $= \dfrac{7000 \times 6}{100} = 420$ Rs.

The transfer therefore increases the income by 120 Rs.

Ex. XLVIII.

1. Find the amount of stock that can be purchased by investing

(1) 4818 Rs. 12 a. in the 4 per cents. at $96\frac{3}{8}$.
(2) 4650 Rs. in the 4 per cents. at 93.
(3) 63000 Rs. in the $5\frac{1}{2}$ per cents. at 10[illegible]
(4) £3380 in the 3 per cents. at $84\frac{1}{2}$.
(5) £9072. 10s. in the 4 per cents. at $95\frac{1}{2}$.
(6) 27000 Rs. in the 6 per cents. at 108.

2. Find what sum will purchase

(1) 10000 Rs. in the 4 per cents. at 97.
(2) 12000 Rs. in the 5 per cents. at 105.
(3) 200 Rs. in the $4\frac{1}{2}$ per cents. at 101.
(4) £600 in the 3 per cents. at 88.
(5) £1800 in the 4 per cents. at 96.
(6) 25000 Rs. in the 6 per cents. at 106.

3. Find the yearly income arising from the investment of

(1) 9637 Rs. 8a. in the 4 per cents. at $96\frac{3}{8}$.
(2) £13950 in the 3 per cents. at 93.
(3) £13520 in the $3\frac{1}{2}$ per cents. at $84\frac{1}{2}$.
(4) 31500 Rs. in the 6 per cents. at 105.
(5) 45362 Rs. 8 a. in the 4 per cents. at $95\frac{1}{2}$.
(6) 13500 Rs. in the 6 per cents. at 108.

4. Find which of the two is the more profitable investment in each of the following Examples :—

(1) The 4 per cents. at 96 and the 5 per cents. at 108.
(2) The 3 per cents. at 84 and the 4 per cents. at 96.
(3) The 4 per cents. at 95 and the 6 per cents. at 115.
(4) The $4\frac{1}{2}$ per cents. at 105 and the 4 per cents. at 96
(5) The $3\frac{1}{2}$ per cents. at 90 and the 4 per cents. at 98.
(6) The 6 per cents. at 112 and the $4\frac{1}{2}$ per cents. at 92.

5. Find the change in income resulting from the transfer of

(1) 15000 Rs. stock from the 4 per cents. at 98 to the 5 per cents. at 105.

(2) 12000 Rs. stock from the 5 per cents. at par to the $4\frac{1}{2}$ per cents. at 90.

(3) £6300 stock from the 3 per cents. at 84 to the 5 per cents. at 105.

6. At what rate per cent. does a person receive interest who invests his capital

(1) In the 4 per cents. at 94?
(2) In the 3 per cents. at 84?
(3) In the $4\frac{1}{2}$ per cents. at 102?
(4) In the 6 per cents. at 112?

7. If 100 Rs. stock in the 4 per cents. can be purchased for 96 Rs. 10 a., for what sum may the same quantity of stock be purchased in the $4\frac{1}{2}$ per cents. with equal advantage?

8. If a person invest 14456 Rs. 4 a. in the 4 per cents. when they are at $96\frac{3}{8}$, what will be his loss of property when they fall to 96?

CHAPTER XIII.

ALLIGATION.

251. DEFS. ALLIGATION is a method of finding the price of a mixture when the prices and quantities of the ingredients are given, or of finding the quantities of the ingredients when the prices of the ingredients and of the mixture are given. In the former case it is called ALLIGATION MEDIAL, and in the latter, ALLIGATION ALTERNATE.

252. In Examples of Alligation Medial, proceed thus:—

RULE. Multiply each price by the corresponding quantity, and divide the sum of the products so obtained by the sum of the quantities. The quotient will be the price of the mixture.

The *reason for the Rule* will be evident from the Example given below.

Ex. Find the price per maund of a mixture of 2 mds. of rice at 4 Rs.. a maund, 3 mds. at 3 Rs. a maund, and 4 mds. at 2 Rs. a maund.

The price of 2 mds. at 4 Rs.. $= 2 \times 4$ or 8 Rs.
................3............3...... $= 3 \times 3$ or 9 ....
................4............2...... $= 4 \times 2$ or 8 ....
$\therefore$ ....9..... of mixture $= 25$ Rs.
and $\therefore$................1.................... $= \frac{25}{9}$ Rs.
$= 2$ Rs. 12a. $5\frac{1}{3}$ pies.

253. The Rule for working out Examples of Alligation Alternate may be deduced thus:—

First let there be only 2 ingredients at prices 12 Rs. and 20 Rs. per maund, to be mixed so that the price of the mixture may be 18 Rs. per maund.

Let $x : y$ or $\frac{x}{y}$ be the ratio of the quantities reqd.

Then by mixing x parts at 12 Rs., and y parts at 20 Rs. we shall have a mixture of $x + y$ parts at 18 Rs.;

$$\therefore \frac{12 \times x + 20 \times y}{x + y} = 18 \text{ (by the preceding Rule)}$$

or, multiplying both sides by $x + y$,

$$12 \times x + 20 \times y = 18 \times x + 18 \times y,$$

or subtracting $12 \times x$ from both sides,

$$20 \times y = 18 \times x - 12 \times x + 18 \times y,$$

or subtracting $18 \times y$ from both sides,

$$20 \times y - 18 \times y = 18 \times x - 12 \times x,$$

$$\text{or } y \times (20 - 18) = x \times (18 - 12),$$

or dividing both sides by $y \times (18 - 12)$,

$$\frac{x}{y} = \frac{20 - 18}{18 - 12};$$

i. e., the quantity of each kind is proportional to the difference between the price of the other and that of the mixture.

Next, let there be several ingredients at prices 12 Rs. 14 Rs., 20 Rs. and 22Rs.

Arrange the prices in the manner shown in the annexed figure, and join the prices of the ingredients in pairs such as 12 and 22, and 14 and 20, so that each pair may consist of one price below and one above the price of the mixture. Then for each pair, find the quantities as in the first case. Thus corresponding to the prices 12 and 22, we have the quantities $22 - 18$ and $18 - 12$, *i.e.*, 4 and 6 respectively; and similarly, corresponding to the prices 14 and 20, the quantities 2 and 4 respectively. And mixing each pair of quantities, we have a mixture of 10 parts, at 18 Rs., whereof 4 parts are at 12 Rs. and 6 at 22 Rs. and another mixture of 6 parts at 18 Rs., whereof 2 parts are at 14 Rs. and 4 at 20 Rs.

$$18 \left| \begin{array}{lc} 12 & 4 \\ 14 & 2 \\ 20 & 4 \\ 22 & 6 \end{array} \right.$$

Now since each of these two mixtures of two ingredients has the given price 18 Rs., a mixture of any quantities of these mixtures will have the same price 18 Rs. Therefore a mixture of the 10 parts of the first mixture and the 6 parts of the second will give a mixture of 16 parts at 18 Rs., whereof 4 parts are at 12 Rs., 2 at 14 Rs., 4 at 20 Rs., and 6 at 22 Rs.

Hence we deduce the following general Rule.

Rule. On one side of a vertical line place the price of the mixture, and on the other, the prices of the ingredients in descending order. Alligate the prices in pairs so that each pair may consist of one price below and one above that of the mixture, and so that every price may be joined with one or more others. Opposite each price, place the difference or the differences between the price or prices with which it is linked and the price of the mixture. The number or the sum of the

numbers opposite to each price will represent the corresponding quantity.

Ex. Find the quantities when their prices are 2 Rs., 4 Rs., 6 Rs., 8 Rs. and 10 Rs. a seer respectively, so that the price of the mixture may be 7 Rs. a seer.

Proceeding by the Rule, we take two modes of linking which are marked I and II.

$$\text{I}\qquad 7\left|\begin{array}{ll} 2 & 3 \\ 4 & 1 \\ 6 & 1 \\ 8 & (3+1) \\ 10 & 5 \end{array}\right. \qquad\qquad \text{II}\qquad 7\left|\begin{array}{ll} 2 & 3 \\ 4 & 1 \\ 6 & 3 \\ 8 & 3 \\ 10 & (5+1) \end{array}\right.$$

In I, we link 2 and 10, 4 and 8, 6 and 8; and opposite to 8 we have two numbers 3 and 1 which are respectively the differences between 7 and the numbers 4 and 6 with which it is linked; and opposite to each of the others, we have one number.

In II, we link 2 and 10, 4 and 8, 6 and 10, and get two numbers opposite to 10.

We can verify each of the results thus :—

$$\frac{2\times3+4\times1+6\times1+8\times4+10\times5}{3+1+1+4+5}=\frac{98}{14}=7;$$

$$\frac{2\times3+4\times1+6\times3+8\times3+10\times6}{3+1+3+3+6}=\frac{112}{16}=7.$$

Thus we see that we can have as many modes of mixing as there are modes of joining the prices two and two according to the Rule.

From the process of *alligating* or joining the prices, this method of operation is called Alligation.

Ex. XLIX.

1. Find the price per maund of a mixture of 3 mds. of sugar at 15 Rs. a maund, 4 mds. at 13 Rs. 8a., and 5 mds. at 11 Rs.

2. Find the price per dram of a mixture of 2 oz. of quinine at 12 Rs. an ounce, 3 oz. at 11 Rs. 8a. an ounce, and 4 oz. at 9Rs. an ounce.

3. Find the price per maund of a mixture of 10 mds. of rice at 4 Rs. a maund, 12 mds. at 4 Rs. 8a., and 8 mds. at 3 Rs. 4a.

4. Find the price per pound of a mixture of 20 lbs. at £1. 10*s.* a pound, 3 qrs. at 18*s.* a pound, and 1 cwt. at £1. 7*s.* a pound.

5. Find the proportion of the ingredients when their prices are 2 Rs. and 4 Rs. per seer, so that the price of the mixture may be 3 Rs. per seer.

6. Find the proportion of the ingredients when their prices are 3 Rs., 5 Rs., and 6 Rs. per maund, so that the price of the mixture may be 4 Rs. per maund.

7. Find the proportion of the ingredients when their prices are 1R. 8a. a seer, 12a. a seer, and 40 Rs. a maund, so that the price of the mixture may be 50 Rs. a maund.

8. The price of an alloy per seer is 14a., and the prices of its ingredients per seer are 12a., and 1 R. 4a. Find the proportion of the ingredients.

9. Find the proportion of the ingredients when their prices are 3 Rs., 4 Rs., 5 Rs., and 6 Rs. per seer, so that the price of the mixture may be 4 Rs. per seer.

CHAPTER XIV.

EXCHANGE.

254. DEFS. The method of converting any sum of money of one country into an equivalent sum of money of another is called EXCHANGE.

The PAR OF EXCHANGE means the *intrinsic* value of a coin of one country as compared with that of another.

By the COURSE OF EXCHANGE is meant the *actual* value at any time of a coin of one country in terms of a coin of another.

Thus, a rupee weighs 180 grs. Troy, whereof $\frac{11}{12}$ths or 165 grs. are pure silver (Art. 159); and a shilling weighs $\frac{1}{66}$ of 1 lb. Troy or $\frac{12 \times 20 \times 24}{66}$ grs. whereof $\frac{37}{40}$ths that is $80\frac{8}{11}$ grs. are pure silver (Art. 142) : so that the quantity of silver contained in a rupee is a little more than what is contained in 2 shillings; and assuming the expense of coinage to be the same in both cases, the intrinsic value of 1R. : that of 1*s*. :: 165 : $80\frac{8}{11}$, *i.e.*, the intrinsic value of 1 R. is a little more than that of 2*s*. Hence at *par* 1 R. ought to exchange for a little more than 2*s*. But actually, the expense of coinage and a variety of other causes produce a variable *course of exchange*, a rupee exchanging sometimes for 2*s*., sometimes for a little more, and sometimes for a little less.

ARBITRATION OR COMPARISON OF EXCHANGES is the method of finding the rate of exchange between the first and the last of a given number of places, when the rates between the first and second, the second and the third &c. of these places are known.

255. We have already in Art. 193 indicated the method of converting the money of one country to that of another. We shall here add a few more Examples.

Ex. 1. A person in Calcutta wishes to remit £6700 to his agent in London. What must he pay when the exchange at 2*s*. $1\frac{1}{8}d$. a rupee?

Let x = sum reqd. in rupees.

Then ∵ 1R. is worth 2*s*. $1\frac{1}{8}d$. or $\frac{67}{32}s$.

and xRs. are.........$6700 \times 20s.$,

$\therefore x : 1 :: 6700 \times 20 : \frac{67}{32}$;

whence $x = \frac{6700 \times 20 \times 32}{67} = 64000$, the no. of rupees reqd.

Ex. 2. A person pays 64000 Rs. for a bill of exchange, at $2s.\ 1\frac{1}{8}d.$ per rupee. What is the amount of the bill?

Let x = amount reqd. in pounds.

Then $\because$ 1 R. is worth $\frac{67}{32}$ s. or £$\frac{67}{640}$

and 64000 Rs. are......£ x,

$\therefore x : \frac{67}{640} :: 64000 : 1$;

whence $x = \dfrac{67 \times 64000}{640} = 6700$ the no of pounds reqd.

Ex. 3. A person has to pay 64000 Rs. for a bill of exchange for £ 6700. What is the course of exchange?

Since 64000 Rs. = £6700

$$\therefore \quad 1 \text{ R.} = £\frac{6700}{64000}$$
$$= £\frac{67}{640}$$
$$= 2s.\ 1\tfrac{1}{8}\ d.$$

Ex. 4. If the exchange between London and Calcutta be at $2s.\ 1d.$ a rupee, and that between Calcutta and New York at 2·24 Rs. a dollar, what is the course of exchange between London and New York?

$$1 \text{ R.} = 2s.\ 1d. = \tfrac{25}{12}\ s.;$$

and 1 dollar = 2·24 Rs.

$$= \frac{224}{100} \times \tfrac{25}{12}s.$$
$$= \tfrac{56}{12}s. = 4s.\ 8d.$$

Ex. L.

1. A person in Calcutta wishes to remit £340 to his agent in London. What must he pay when the exchange is at $2s.\ 1\frac{1}{2}d.$ a rupee?

2. What is the value in Indian money of £500 when a rupee is worth $1s.\ 8d.$?

3. A person has to pay 4500 Rs. 8 a. for a debt due to a merchant in London, when 1 R. = $1s.\ 9d.$ What is the amount of the debt in English money?

4. A person has to pay Rs. 6000 for a debt of £550 due to a creditor in London. What is the value of a rupee in English money?

5. When the exchange between England and India is $1s.\ 10d.$ per rupee, and between England and St. Petersburg,

2*s* 9*d* per ruble, what is the course of exchange between India and St Petersburg?

6. If the exchange between England and India, and between England and St. Petersburg be as in the preceding Example, and that between India and St. Petersburg be $1\frac{1}{3}$ Rs per ruble, is it more advantageous to a person in Calcutta to remit money directly to London, or circuitously through St. Petersburg?

CHAPTER XV.

SQUARE ROOT. CUBE ROOT.

SEC. I. SQUARE ROOT.

256 DEFS. The SQUARE of a given number is the product of that number multiplied by itself. It is also called the second power of that number (Art. 42), and is denoted by placing the figure 2 above the number a little to its right. Thus $9^2 = 9 \times 9 = 81$.

The SQUARE ROOT of a given number is the number whose square is equal to the given number. It is denoted by the sign $\sqrt{}$ placed before the given number.

Thus $\sqrt{16} = 4$.

257. Since

$$\begin{aligned} \sqrt{1} &= 1 \\ \sqrt{100} &= 10 \\ \sqrt{10000} &= 100 \\ \sqrt{1000000} &= 1000 \\ \&c. &= \&c., \end{aligned}$$

∴ the square root of any number
between 1 and 100 must consist of 1 figure
.........100 and 10000..................2 figures
.....10000 and 1000000..............3.........
and so on.

Hence if we place a dot over the figure in the units' place of any given number, and thence over every second figure to the left, the number of dots will be equal to the number of figures in the integral part of the root. Thus the number 2436 with the dots will stand thus, $2\dot{4}3\dot{6}$, shewing that there are two figures in the integral part of the root.

258. *To find the square root of a given number.*

By pointing the given number as in Art. 257, we can ascertain the number of figures in the integral part of the root. Now let us see how the root itself is to be found out.

Take any number 48.

Then $48^2 = 2304$

and also $= (40+8)^2 = (40+8) \times (40+8)$

$$\begin{aligned} &= 40 \times (40+8) + 8 \times (40+8) \\ &= 40^2 + 8 \times 40 + 8 \times 40 + 8^2 \\ &= 40^2 + 2 \times 8 \times 40 + 8^2 \end{aligned}$$

In this last form in which 48^2 can be written, we see how the parts of the root enter into the composition of the power. Now let us see how we can obtain 48 or $40+8$ from the last expression. The first part of the root, *i.e.*, 40 is the square root of 40^2. Subtracting 40^2 from the given number, we have $2 \times 8 \times 40 + 8^2$ left. Now if we divide $2 \times 8 \times 40$ by 2×40, we get 8, the second figure in the root. And multiplying $2 \times 40 + 8$ by 8, we get $2 \times 40 \times 8 + 8^2$; and subtracting this product from $2 \times 8 \times 40 + 8^2$, the portion of the given number left after deducting 40^2, we have nothing more left.

If the root be a number consisting of more figures than two, for instance, if it be 483, it may be written as $480+3$, and its square as $480^2 + 2 \times 3 \times 480 + 3^2$. Here having found 4 and 8 as before, we may find the third figure 3 in the same way as above, by supposing 480 to stand in the place of the number 40 in the first case.

The process may be stated thus :—

$$\begin{array}{r|l} & 40^2 + 2 \times 8 \times 40 + 8^2 \;(40+8 \\ & 40^2 \\ \hline 2 \times 40 + 8 & 2 \times 8 \times 40 + 8^2 \\ & 2 \times 8 \times 40 + 8^2 \\ \hline \end{array}$$

or thus :—

$$\begin{array}{r|l} & 1{,}600 + 640 + 64 \;(40+8 \\ & 1{,}600 \\ \hline 80 + 8 & 640 + 64 \\ & 640 + 64 \\ \hline \end{array}$$

or thus (omitting cyphers, and representing the given number as one number without breaking it into parts) :—

$$\begin{array}{r|l} & 2304 \;(48 \\ & 16 \\ \hline 88 & 704 \\ & 704 \\ \hline \end{array}$$

Hence we deduce the following Rule :—

RULE. Place a dot over the figure in the units' place of the given number, and thence over every second figure to the left, thus dividing the number into several periods.

Find the greatest number whose square is not greater than the first period at the left: this will be the figure in the highest place in the root required. Place it in the form of a quotient, subtract its square from the first period, and to the remainder, if any, annex the figures in the next period in the given number. Divide the number thus obtained, omitting its last figure, by twice the part of the root already obtained, annex the quotient to the part of the root already obtained and to the divisor, and multiply the resulting divisor by the quotient, and subtract the product from the number formed by the first remainder and the second period. To the remainder annex the next period, and proceed as before. Repeat this process for each successive period.

Ex. Extract the square root of 12769.

$$
\begin{array}{r|l}
 & \dot{1}2\dot{7}6\dot{9} \;(\,113 \\
 & 1 \\ \hline
21 & 27 \\
 & 21 \\ \hline
223 & 669 \\
 & 669 \\ \hline
\end{array}
$$

259. Since $\sqrt{24\cdot66} = \sqrt{\dfrac{2466}{100}} = \dfrac{\sqrt{2466}}{10}$,

$$\sqrt{1\cdot634} = \sqrt{\frac{1634}{1000}} = \sqrt{\frac{16340}{10000}} = \frac{\sqrt{16340}}{100},$$

$$\sqrt{.0066} = \sqrt{\frac{.66}{10000}} = \frac{\sqrt{66}}{100},$$

&c. = &c. = &c.;

∴ it follows that in extracting the square root of a decimal with or without an integral part, if we make the number of decimal places even by annexing a cypher when necessary, and extract the square root of the resulting number regarded as an integer, and in the root thus obtained, point off a number of decimal places equal to half the number of those in the square, we shall obtain the true root.

$$\text{Again}\quad \sqrt{12} = \sqrt{\frac{1200}{100}} = \frac{\sqrt{1200}}{10},$$
$$\sqrt{1\cdot33} = \sqrt{\frac{1330000}{10000}} = \frac{\sqrt{1330000}}{100},$$
$$\sqrt{1\cdot25} = \sqrt{\frac{12500}{10000}} = \frac{\sqrt{12500}}{100},$$
$$\&c. = \&c. = \&c.;$$

∴ where the operation does not terminate, we can carry it on to any length by annexing periods of two cyphers to the given number and obtaining one decimal place in the root corresponding to every such period.

Ex. 1. Extract the square root of 6·25.

```
       6·25 (2·5
       4
  45 | 225
     | 225
```

Ex. 2. Extract the square root of 2 to 3 places of decimals.

```
        2·000000 ( 1·414
        1
   24 | 100
      |  96
  281    | 400
         | 281
  2824   | 11900
         | 11296
             604
```

260. The square root of a fraction may be found thus.—

Let it be required to find the square root of $\frac{3}{5}$.

$$\text{Now}\qquad \sqrt{\tfrac{3}{5}} = \sqrt{\frac{3\times5}{5\times5}} = \frac{\sqrt{3\times5}}{5} = \frac{\sqrt{15}}{5}.$$

So that the square root of $\sqrt{\frac{3}{5}}$ is found by finding that of 15 and dividing the result by 5.

Or $\sqrt{\frac{3}{5}}$ may be found by reducing $\frac{3}{5}$ to a decimal, and then extracting the square root of that decimal by the method indicated in Art. 259.

261. Since the number of square units in a square = the square of the number of linear units in a side,
$\therefore$ the number of linear units in a side of a square = the square root of the number of square units in the area.

Ex. A square plot of land measures 1 acre. Find the length of its side in yards.

$$1 \text{ ac.} = 4 \times 40 \times \frac{121}{4} \text{sq. yds.} = 4840 \text{ sq. yds.},$$

$\therefore$ the length reqd. = $\sqrt{4840}$ yds.

Now $\sqrt{4840} = 69{\cdot}57$...as shewn below :

```
            . .  . .
         4840·0000  ( 69·57
         36
   129   1240
         1161
  1385     7900
           6925
 13907      97500
            97349
              151
```

$\therefore$ the length reqd. = 69·57...... yds. approximately.

The result can be verified thus :—

The area of the square whose side is 69·57 yds.
$= 69{\cdot}57 \times 69{\cdot}57$ sq. yds.
$= 4839{\cdot}9849$ sq. yds.
which differs from 1 ac. by $(4840 - 4839{\cdot}9849)$ sq. yds., *i.e.*, by ·0151 sq. yd., *i.e.*, by less than $\frac{1}{60}$ sq. yd.

If we take a few more decimal places in the root, the difference will become smaller still. In this way, we can find the side to any degree of approximation.

Ex. LI.

1. Find the square roots of
   (1) 441; 961; 9801; 7921; 12321; 49284.
   (2) 1681; 2601; 6241; 4761; 110889.
   (3) 625; 1225; 2025; 3025; 4225; 5625.
   (4) 7225; 9025; 15129; 54756; 18225.
   (5) 1522756; 1234321; 4937284; 1002001.
   (6) 11108889; 4080400; 25010001.

2. Extract (to 4 places of decimals where the root does not terminate) the square roots of

(1) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
(2) 11, 12, 13, 14, 99, 77, 66.
(3) 1·2, 2·3, 3·4, 2·34, 1·44.
(4) ·1, ·002, ·0004, 100·001, ·9.
(5) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$.
(6) $\frac{2}{3}$, $\frac{3}{11}$, $\frac{25}{144}$, $\frac{36}{361}$, $\frac{25}{225}$.

3. A square plot of land contains 50 bghs. Find the length of its side.

4. A rectangular plot of land whose length is equal to its breadth, contains 5 acres. What is its length?

5. A rectangular piece of land whose length is equal to its breadth, contains as much land as another piece, which is 5 bghs. long and 2 bghs. broad. Find the length of the former.

6. A rectangular piece of land whose length is double of its breadth, measures 800 ft. along its length. What must be the length of another field whose breadth is equal to its length, in order that it may contain the same area?

7. The perimeter of a rectangular field whose breadth is half of its length, is 6 bghs. Find the perimeter of a square plot of land containing the same area.

8. Two square plots of land contain respectively 2 square miles and 10 acres. Find the difference between their sides.

SEC. II. CUBE ROOT.

262. Defs. The Cube of a number is the continued product of that number repeated as a factor thrice. This product is also called the third power of that number (Art. 42), and is denoted by placing the figure 3 above the number a little to its right.

Thus $8^3 = 8 \times 8 \times 8 = 512$.

The Cube Root of a given number is the number whose cube is equal to the given number. It is denoted by the sign $\sqrt[3]{}$ placed before the given number.

Thus $\sqrt[3]{125} = 5$

263. Since

$$\sqrt[3]{1} = 1$$
$$\sqrt[3]{1000} = 10$$
$$\sqrt[3]{1000000} = 100$$
$$\sqrt[3]{1000000000} = 1000$$
&c. = &c.

∴ the cube root of any number

between 1 and 1000 must consist of 1 figure
......1000 and 1000000 ....2 figures
...1000000 and 1000000000...............3........
and so on.

Hence if we place a dot over the figure in the units' place of any given number, and thence over every third figure to the left, the number of dots will be equal to the number of figures in the integral part of the root. Thus the number 2346785 with the dots will stand thus, $\dot{2}34\dot{6}78\dot{5}$, shewing that there are three figures in the integral part of the root.

264. *To find the cube root of a given number.*

By pointing the given number as in Art. 263, we can ascertain the number of figures in the integral part of the root Now let us see how the root itself is to be found out.

Take any number 18.

Then $18^3 = 5832$

and also $= (10+8)^3 = (10+8) \times (10+8) \times (10+8)$
$$= (10^2 + 2 \times 10 \times 8 + 8^2) \times (10+8)$$
$$= 10 \times (10^2 + 2 \times 10 \times 8 + 8^2)$$
$$+ 8 \times (10^2 + 2 \times 10 \times 8 + 8^2)$$
$$= 10^3 + 2 \times 10^2 \times 8 + 8^2 \times 10$$
$$+ 8 \times 10^2 + 2 \times 8^2 \times 10 + 8^3$$
$$= 10^3 + 3 \times 10^2 \times 8 + 3 \times 10 \times 8^2 + 8^3.$$

In this last form in which 18^3 can be written, we see how the parts of the root enter into the composition of the power. Now let us see how we can obtain 18 or $10+8$ from the last expression. The first part of the root, *i.e.*, 10, is the cube root of 10^3. Subtracting 10^3 from the given number, we have $3 \times 10^2 \times 8 + 3 \times 10 + 8^2 + 8^3$ left. Now if we divide

$3\times10^2\times8$ by 3×10^2, we get 8, the second figure in the root. And multiplying $3\times10^2+3\times10\times8+8^2$ by 8, we get $3\times10^2\times8+3\times10\times8^2+8^3$, and subtracting this product from $3\times10^2\times8+3\times10\times8^2+8^3$, the portion of the given number left after deducting 10^3, we have nothing more left.*

If the root be a number consisting of more figures than two, for instance, if it be 185, it may be written as $180+5$, and its cube as $180^3+3\times180^2\times5+3\times180\times5^2+5^3$. Here having found 1 and 8 as before, we can find the third figure 5 in the same way as above, by supposing 180 to stand in the place of the number 10 in the first case.

The process may be stated thus :—

$$
\begin{array}{l|l}
 & 10^3+3\times10^2\times8+3\times10\times8^2+8^3\ (10+8 \\
 & 10^3 \\
\hline
3\times10^2 & 3\times10^2\times8+3\times10\times8^2+8^3 \\
\quad +3\times10\times8 & \\
\quad\quad +8^2 & \\
\hline
3\times10^2+3\times10\times8+8^2 & 3\times10^2\times8+3\times10\times8^2+8^3 \\
\hline
\end{array}
$$

or thus :—

$$
\begin{array}{r|l}
 & 1000+2400+1920+512\ (10+8 \\
 & 1000 \\
\hline
3\times10^2=300 & 2400+1920+512 \\
3\times10\times8\ =240 & \\
8^2=\ \ 64 & \\
\hline
300+240+64 & 2400+1920+512 \\
\hline
\end{array}
$$

or thus (omitting cyphers and representing the given number as one number without breaking it into parts) :—

$$
\begin{array}{r|l}
 & 5832\ (18 \\
 & 1 \\
 & 4832 \\
3\times10^2=300 & \\
3\times10\times8=240 & \\
8^2=\ \ 64 & \\
\hline
604 & 4832 \\
\end{array}
$$

Hence we deduce the following Rule:—

RULE. Place a dot over the figure in the units' place of the given number, and thence over every third figure to the left, thus dividing the number into several periods.

Find the greatest number whose cube is not greater than the first period at the left : this will be the figure in the highest place in the root required. Place it in the form of a quotient, subtract its cube from the first period, and to the remainder annex the figures in the next period in the given number. Divide the number thus obtained by thrice the square of the part of the root already obtained (regarded as so many tens), in order to find by trial the greatest number such that the product of that number multiplied by the value of 3 × the square of the part of the root already obtained (regarded as so many tens) + 3 × the part of the root already obtained (regarded as so many tens) × that number + the square of that number is not greater than the dividend under consideration. That number will be the next figure in the root. Subtract the product above mentioned from the number composed of the first remainder and the second period. To the remainder annex the third period if any, and proceed as before. Repeat this process for each successive period.

Ex Find the cube root of 6859

$$
\begin{array}{rl}
 & \dot{6}85\dot{9}\ (19 \\
 & 1 \\
3 \times 10^2 = 300 & \overline{5859} \\
3 \times 10 \times 9 = 270 & \\
9^2 = 81 & \\
\hline
651 & 5859 \\
\end{array}
$$

265. Since $\sqrt[3]{18\cdot775} = \sqrt[3]{\dfrac{18775}{1000}} = \dfrac{\sqrt[3]{18775}}{10}$

$$\sqrt[3]{2\cdot5786} = \sqrt[3]{\frac{2578600}{1000000}} = \frac{\sqrt[3]{2578600}}{100}$$

$$\sqrt[3]{\cdot006781} = \sqrt[3]{\frac{6781}{1000000}} = \frac{\sqrt[3]{6781}}{100};$$

$\therefore$ it follows that in extracting the cube root of a decimal with or without an integral part, if we make the number of decimal places equal to 3 or a multiple of 3 by affixing cyphers when necessary, and extract the cube root of the resulting number regarded as an integer, and in the root thus obtained, point off a number of decimal places equal to one-third of the number of those in the cube, we shall obtain the true root.

$$\text{Again } \because \sqrt[3]{13} = \sqrt[3]{\frac{13000}{1000}} = \frac{\sqrt[3]{13000}}{10};$$

$$\sqrt[3]{14 \cdot 4} = \sqrt[3]{\frac{14400000000}{1000000000}} = \frac{\sqrt[3]{14400000000}}{1000};$$

$$\sqrt[3]{1 \cdot 26} = \sqrt[3]{\frac{1260000}{1000000}} = \frac{\sqrt[3]{1260000}}{100}$$

$$\&c. = \&c. = \&c.$$

$\therefore$ when the operation does not terminate, we can carry it on to any length by annexing periods of three cyphers to the given number, and obtaining one decimal place in the root corresponding to every such period.

Ex. 1. Extract the cube root of ·12500

$$\sqrt[3]{\cdot 12500} = \sqrt[3]{\frac{125000}{1000000}} = \frac{\sqrt[3]{125000}}{100}$$

$$\begin{array}{l} 125\dot{0}0\dot{0} \;(\; 50 \\ \underline{125} \end{array}$$

$\therefore$ ·50 is the root reqd.

Ex. 2. Find the cube root of 4 to 2 places of decimals.

$$\begin{array}{rl|l}
 & & 4 \cdot 000000 \;(\; 1 \cdot 58 \\
 & & \underline{1} \\
3 \times 10^2 = & 300 & 3000 \\
3 \times 10 \times 5 = & 150 & \\
5^2 = & 25 & \\
 & \overline{475} & \underline{2375} \\
 & & 625000 \\
3 \times 150^2 = & 67500 & \\
3 \times 150 \times 8 = & 3600 & \\
8^2 = & 64 & \\
 & \overline{71164} & \underline{569312} \\
 & & 55688
\end{array}$$

266. The cube root of a fraction may be found thus :—

Take as an example, $\sqrt[3]{\frac{2}{3}}$.

Now, $\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{2 \times 3 \times 3}{3 \times 3 \times 3}} = \frac{\sqrt[3]{18}}{3}$.

So that the cube root of $\frac{2}{3}$ is found by finding that of 18, and dividing the result by 3.

Or $\sqrt[3]{\frac{2}{3}}$ may be found by reducing $\frac{2}{3}$ to a decimal, and extracting the cube root of that decimal by the method indicated in Art. 265.

267. Since the number of solid units in a cube = the cube of the number of linear units in an edge of it,

∴ the number of linear units in an edge of a cube = the cube root of the number of solid units in the cube.

Ex. A cubical room contains 1728 cub. feet. Find its height.

Height reqd. in feet $= \sqrt[3]{1728}$
$= 12.$

Ex. LII.

1. Find cube roots of
   (1) 1331 ; 1728 ; 2197 ; 2744.
   (2) 3375 ; 4096 ; 4913 ; 5832 ; 6859.
   (3) 15625 ; 1367631 ; 10941048.
2. Extract (to 2 places of decimals where the root does not terminate) the cube roots of
   (1) 1 ; 2 ; 3 ; 4 ; 5 ; 6 ; 7 ; 8 ; 9.
   (2) $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$; $\frac{1}{5}$; $\frac{1}{6}$; $\frac{1}{7}$; $\frac{1}{8}$; $\frac{1}{9}$.
   (3) ·1 ; ·2 ; ·3 ; ·001 ; ·01.
   (4) 123·456 ; 166·375 ; 287·496.
3. Find the length of a cubical room which contains 2744 cub. ft.
4. A cube contains 6859 cub. in. Find its edge.

ANSWERS TO THE EXAMPLES.

Ex. I.

1.

(1) 10 ; 12 ; 15 ; 19 ; 28 ; 44 ; 56 ; 61; 84 ; 92.

(2) 101 ; 110 ; 154 ; 300 ; 405 ; 560 ; 774.

(3) 1001 ; 2051 ; 3263 ; 4000 ; 5500 ; 6780.

(4) 100001 ; 200300 ; 306709 ; 456004 ; 567432.

(5) 2000001 ; 3000029 ; 4000560 ; 5600074 ; 6754321.

(6) 3000000000000 ; 4000000000005 ; 5000000000708 ; 7000913579135.

(7) 1900000000000000000000 ; 2000000000000000000024 ; 3100000055670982752O.

(8) 100001 ; 203003 ; 561720 ; 1530612.

(9) 20000002 ; 30507009 ; 56432178.

(10) 2165016718.

2.

(1) Eighteen ; twenty ; thirty-seven ; fifty-eight ; sixty-nine ; eighty-five ; ninety-seven.

(2) Two hundred and three ; three hundred and forty ; four hundred and fifty-six ; six hundred and ninety ; seven hundred and eight ; nine hundred and ninety-one.

(3) One thousand and nine ; two thousand and twenty-nine ; three thousand six hundred and ninety ; four thousand eight hundred and sixty-two.

(4) One hundred and two thousand and thirty ; two hundred and thirty thousand four hundred and fifty ; three hundred thousand and four ; seven hundred and forty-five thousand six hundred and twenty-one.

(5) One hundred and twenty-three millions, four hundred and fifty-six thousand seven hundred and eighty-nine ; nine hundred and eighty-seven millions, six hundred and fifty-four thousand three hundred and twenty-one ; one hundred and two millions, thirty thousand four hundred and five.

(6) Two thousand four hundred and sixty-eight millions, one hundred and one thousand two hundred and fourteen ; two hundred and forty-eight thousand one hundred and sixty-three millions, two hundred and sixty-four thousand one hundred and twenty-eight.

(7) Fifty billions, one hundred thousand two hundred millions, three hundred thousand four hundred; thirty-six thousand nine hundred and twelve billions, one hundred and fifty-one thousand eight hundred and twenty-one millions, two hundred and forty-two thousand seven hundred and thirty.

(8) Two trillions, three hundred and five thousand eight hundred forty-three billions, eight thousand one hundred and thirty-nine millions, nine hundred and fifty-two thousand one hundred and twenty-eight; one hundred and thirty-seven thousand four hundred and thirty-eight millions, six hundred and ninety-one thousand three hundred and twenty-eight.

3. One lac two thousand and thirty; two lacs thirty thousand four hundred and fifty; three lacs and four; seven lacs forty-five thousand six hundred and twenty-one.

Twelve crores thirty-four lacs fifty-six thousand seven hundred and eighty-nine; ninety-eight crores seventy-six lacs fifty-four thousand three hundred and twenty-one; ten crores twenty lacs thirty thousand four hundred and five.

4. 10+8 ; 20 ; 30+7 ; 50+8 ; 60+9 ; 80+5 ; 90+7.
200+3 ; 300+40 ; 400+50+6 ; 600+90 ; 700+8 ; 900+90+1.
1000+9 ; 2000+20+9 ; 3000+600+90 ; 4000+800+60+2.

5. (1) XXV ; XXXIII ; XLVI ; LXXXVII ; XCIX.
(2) CI ; CCXX ; CCCXIV ; DXVI ; CMXCIX.
(3) MI ; MDCCCLVI ; MDCCCLXIV.

6. (1) 27 ; 34 ; 45 ; 46.
(2) 99 ; 301 ; 1040 ; 650.
(3) 1856 ; 1582 ; 1009.

Ex. II.

(1) 45. (2) 135. (3) 225. (4) 459.
(5) 29191 (6) 3330. (7) 2643. (8) 3108.
(9) 2997. (10) 46998. (11) 25908. (12) 3053.
(13) 37170. (14) 34655782. (15) 4327338. (16) 98549.
(17) 2420346420. (18) 1607804.

2. 6221449083221. 3. 586468087. 4. 14805 ; 6840.
5. 1269184 ; 295424. 6. 546. 7. 280 ; 92.
8. 171 ; 231 ; 256 ; 512 ; 360 ; 1530.

Ex. III.

1. (1) 6. (2) 9. (3) 10. (4) 11.
(5) 18. (6) 78. (7) 669. (8) 10.
(9) 63. (10) 1. (11) 178882.
(12) 433430. (13) 90909 (14) 164421.
(15) 90909. (16) 801244. (17) 71533517.
(18) 864197532. (19) 9999 (20) 10000998.

2. (1) 4500. (2) 495000. (3) 59700000.

3. 22 ; 45 ; 64 ; 120 ; 129 ; 110 ; 1284 ; 11112 ; 133334 ; 49995.

4. 2000000 ; 200000.

5. (1) 10. (2) 10. (3) 10. (4) 4. (5) 3.
(6) 8. (7) 6. (8) 7. (9) 7. (10) 5.

6. 0 ; 9 ; 4 ; 332 ; 1018 ; 4 ; 544 ; 261 ; 233 ; 140.

7. (1) 167. (2) 12.

8. 3 ; 132 ; 231 ; 139 ; 422 ; 2355.

Ex. IV.

. (1) 492 ; 615 ; 738 ; 984 ; 1107.
(2) 3192 ; 3648 ; 4104 ; 4560 ; 5016.
(3) 2367 ; 4734 ; 7101 ; 9468 ; 11835.
(4) 617283945 ; 1234567890 ; 1851851835 ; 2469135780 ; 3086419725.
(5) 7901234568 ; 11851851852 ; 15802469136 ; 19753086420 ; 23703703704.
(6) 56447784 ; 80958006 ; 57561885.
(7) 97406784 ; 121851072 ; 98517888.
(8) 405811215 ; 505009512 ; 450901350.
(9) 3276941063 ; 3677951156 ; 3605040230.
(10) 82519021020 ; 245939043040 ; 409359065060.
(11) 14446089217728 ; 75640328065008.
(12) 121932631112635269 ; 13411358024859.
(13) 109826000000 ; 82678000000 ; 55530000000.
(14) 516760458000000 ; 688741400000.
(15) 4508515660000 ; 5890616550000.

(1) 1975296. (2) 18172480. (3) 36344960.
(4) 126419200. (5) 5598842958. (6) 3258960.

3. (1) 362880. (2) 46080. (3) 29160. (4) 19019.
4. (1) 285. (2) 2025. (3) 2109375.
5. 1000000000000 ; 240000000000000.
6. 99990000000000 ; 199980000000000.
7. 891000000000.

Ex. V.

1. (1) 617 ; 411, rem. 1 ; 308, rem. 2 ; 246, rem. 4.
(2) 1152, 864 ; 691, rem. 1 ; 576.
(3) 1185, rem. 3 ; 946, rem. 2 ; 811, rem. 1, 709, rem. 6.
(4) 11272, rem. 6 ; 9863, rem. 6 ; 8767, rem. 7 ; 7891.
(5) 24691357, rem. 4 ; 12345678, rem. 9 ; 8230452, rem. 9 ; 6172839, rem. 9 ; 4938271, rem. 14.
(6) 123456790, rem. 1 ; 82304526, rem. 9 ; 61728395 rem. 1 ; 49382716, rem. 1 ; 41152263 ; 1130, rem. 9.
(7) 271, rem. 213 ; 189, rem. 183 ; 266, rem. 99.
(8) 156, rem. 372 ; 125, rem 81 ; 154, rem, 564.
(9) 2474, rem. 33 ; 1988, rem. 51 ; 2226 rem. 303.
(10) 5006, rem. 753 ; 4461, rem. 19 ; 4551, rem 217.
(11) 79314, rem. 721 ; 26612, rem. 521 ; 15988, rem. 1721.
(12) 99720, rem. 10128 ; 19045, rem. 5103
(13) 8000051200, rem. 65145 ; 8000000, rem. 9012345.
(14) 1386, rem. 4600 ; 1841, rem. 5300 ; 2742, rem. 1000.
(15) 900090009, rem. 1 ; 90009000, rem. 10000 ; 9000900, rem 10000.
(16) 100010001 ; 10001000, rem. 1111 ; 11112222, rem. 3333 ; 1111222, rem. 23331.

2. (1) 1728 ; 1152 ; 864 ; 691 rem. 1 ; 576.
(2) 3394, rem. 3 ; 1697, rem. 3 ; 1131, rem. 7 ; 848, rem. 11.
(3) 500000 ; 333333, rem. 1 ; 250000 ; 200000 ; 166666, rem. 4 ; 142859, rem. 1.
(4) 13888888, rem. 7 ; 12345679 ; 11111111, rem. 1 ; 10101010, rem. 1 ; 9259259, rem. 3.
(5) 170940, rem. 2 ; 158730, rem. 2 ; 148148, rem. 2 ; 138888, rem. 14 ; 130718, rem. 16 ; 123456, rem. 14.
(6) 1754385, rem. 18 ; 1666666, rem. 13 ; 1587301, rem. 12 ; 151515, rem. 11 ; 1388888, rem. 21.

(7) 2739197559, rem. 22 ; 2656191573, rem. 1; 2578068291, rem. 16 ; 2504409197, rem. 15.

(8) 587643, rem. 6 ; 560932, rem. 4 ; 514187; rem. 36 ; 503694, rem. 6 ; 483941, rem. 21.

3. 99, rem 99902 ; 9890, rem. 209.

4. 1, rem. 2000,000.

5. 111111, rem. 100000 ; 110000.

Ex. VI.

1. (1) 6 ; 16 ; 12 ; 16 ; 14 ; 19 ; 19 ; 23.
   (2) 12 ; 19 ; 25 ; 16 ; 19 ; 17 ; 19 ; 19.
   (3) 15 ; 8; 4 ; 11 ; 17 ; 289 ; 121 ; 361.
   (4) 38 ; 34 ; 28 ; 43.
   (5) 291 : 237 ; 438 ; 213.
   (6) 202 ; 321 ; 453 ; 724.
   (7) 12 ; 18 ; 15 ; 863.
   (8) 4 ; 4 ; 6.
   (9) 9 ; 33.
2. (1) 6. (2) 12. (3) 5. (4) 36. (5) 19. (6) 66.
3. (1) 108 ; 42 ; 126 ; 126 ; 234 ; 2185 ; 595 ; 931.
   (2) 234 ; 858 ; 225 ; 2023 ; 10830 ; 1104.
   (3) 9372 ; 6636 ; 15554.
   (4) 1999998 ; 1294125.
   (5) 13548070123626141.
4. (1) 2520. (2) 45045. (3) 1680. (4) 6350400. (5) 4200. (6) 360.
   (7) 1085040. (8) 803440. (9) 114578640. (10) 139230. (11) 457 200.
   (12) 118800.
5. (1) $2\times2\times23$; $2\times2\times3\times3\times3$; 3×37 ; 7×17 ; $3\times3\times19$; $3\times3\times3\times7$.
   (2) $2\times2\times3\times17$; $2\times2\times2\times3\times3\times3$; $2\times2\times5\times13$; 17×17 ; $2\times2\times2\times2\times19$.
   (3) $2\times2\times3\times3\times3\times3$; 19×19 ; $2\times3\times67$; 13×31 ; $2\times2\times101$.
   (4) $2\times7\times29$; 11×37 ; 5×83 ; $2\times2\times2\times2\times2\times2\times2\times2\times2$; $2\times3\times103$.
   (5) $3\times11\times23$; 11×71 ; $2\times5\times79$; $2\times2\times3\times3\times23$; $3\times3\times3\times5\times7$.

Ex. VII.

I.

3. 20030040. 5. A has 3, B, 6, and C, 9. 6. 9.
7. 130. 8. 12 in the 1st, 35 in the 2nd, 32 in the 3rd.

II.

1. 20336. 5. 26 in the 1st, 36 in the 2nd, 67 in the 3rd.
6. Rs. 5. 7. Rs. 425. 8. 78.

III.

5. 16200. 6. Rs. 320; Rs. 1920. 7. 144.
8. 64,1024.

IV.

5. $101\frac{7}{13}$ 6. Rs. 5. 7. Rs. 13. 8. 12; 144

V.

5. He can distribute the amount among 2, 3, 4, 6, 8, or 12 men.
6. 12; 5, 7. 60. 8. 2520.

VI.

3. 2. 4. 21. 5. 147. 6. 179. 7. 24, rem. 8.
8. 1.

VII.

3. 121. 4. 453.
5. The price of the horse is Rs. 310; that of the carriage, 1550 Rs.
6. 300Rs. and 550Rs. 7. 12 Rs. 8. 300 Rs.

VIII.

3. 6032Rs. 4. 12. 5. 19; 11286. 6. 15 years
7. 20400. 8. 12.

Ex. VIII.

1. (1) $\frac{10}{2}, \frac{15}{3}, \frac{20}{4}$. (2) $\frac{27}{3}, \frac{36}{4}, \frac{45}{5}$.
(3) $\frac{42}{6}, \frac{49}{7}, \frac{56}{8}$. (4) $\frac{24}{3}, \frac{40}{5}, \frac{56}{7}$.
(5) $\frac{176}{11}, \frac{192}{12}, \frac{208}{13}$. (6) $\frac{100}{4}, \frac{200}{8}, \frac{400}{16}$.

2. (1) 2. (2) 3. (3) 3. (4) 4. (5) 2. (6) 11.
(7) 32. (8) 19. (9) 36. (10) 144 (11) 6. (12) 3.

Ex. IX.

1. (1) $1\frac{1}{2}$. (2) $1\frac{2}{3}$. (3) 2. (4) $1\frac{1}{15}$. (5) $2\frac{4}{7}$
(6) 8. (7) $7\frac{12}{13}$. (8) $23\frac{2}{13}$. (9) 38. (10) $2\frac{11}{15}$.
(11) $14\frac{5}{56}$. (12) $10\frac{10}{99}$.

2. (1) $\frac{4}{3}$. (2) $\frac{17}{5}$. (3) $\frac{38}{7}$. (4) $\frac{89}{10}$
(5) $\frac{143}{12}$ (6) $\frac{256}{19}$. (7) $\frac{496}{17}$. (8) $\frac{157}{8}$
(9) $\frac{183}{11}$ (10) $\frac{3891}{32}$ (11) $\frac{1110}{11}$
(12) $\frac{100001}{100}$.

3. (1) $\frac{1}{4}$. (2) $\frac{1}{6}$. (3) $\frac{3}{25}$. (4) $\frac{1}{15}$.
(5) $\frac{9}{8}$. (6) $\frac{11}{10}$. (7) $\frac{1}{4}$. (8) $\frac{1}{2}$.
(9) $\frac{5}{8}$. (10) $\frac{1}{5}$. (11) $\frac{2}{9}$. (12) $\frac{23}{5}$.
(13) $\frac{189}{935}$. (14) $\frac{1}{10}$. (15) $\frac{1}{9}$. (16) 5.
(17) $\frac{42517}{53248}$. (18) $\frac{4953}{1247}$.

4. (1) $\frac{32}{45}$. (2) $\frac{2}{5}$. (3) $\frac{9}{4}$. (4) $\frac{13}{7}$.
(5) $\frac{20}{3}$ (6) $\frac{5}{4}$ (7) 2. (8) 10.
(9) $\frac{1}{144}$ (10) $\frac{5}{6}$ (11) 5. (12) 15.
(13) $\frac{1}{4}$. (14) $\frac{16}{7}$. (15) $\frac{9}{4}$. (16) 1.

5. (1) $\frac{3}{5}$. (2) $\frac{5}{11}$. (3) $\frac{4}{13}$. (4) $\frac{3}{7}$.
(5) $\frac{11}{90}$. (6) $\frac{19}{24}$. (7) $\frac{8}{9}$. (8) $\frac{5}{21}$.
(9) $\frac{13}{21}$. (10) $\frac{3}{4}$. (11) $\frac{3}{4}$. (12) $\frac{2}{3}$.
(13) $\frac{41}{152}$. (14) $\frac{263}{357}$. (15) $\frac{5}{37}$. (16) $\frac{41}{107}$.
(17) $\frac{4}{107}$. (18) $\frac{31}{105}$. (19) $\frac{16}{45}$. (20) $\frac{16}{13}$.

6. (1) $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$. (2) $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$. (3) $\frac{15}{30}$, $\frac{16}{30}$, $\frac{6}{30}$
(4) $\frac{30}{36}$, $\frac{8}{36}$, $\frac{21}{36}$. (5) $\frac{168}{630}$, $\frac{175}{630}$, $\frac{180}{630}$. (6) $\frac{5}{15}$, $\frac{4}{15}$, $\frac{4}{15}$,
(7) $\frac{27}{72}$, $\frac{32}{72}$, $\frac{36}{72}$, $\frac{38}{72}$. (8) $\frac{168}{504}$, $\frac{216}{504}$, $\frac{252}{504}$, $\frac{280}{504}$.
(9) $\frac{1683}{1716}$, $\frac{1612}{1716}$, $\frac{1599}{1716}$ $\frac{132}{1716}$.
(10) $\frac{672}{840}$, $\frac{700}{840}$, $\frac{720}{840}$, $\frac{735}{840}$.

(11) $\frac{1092}{1638}$, $\frac{910}{1638}$, $\frac{882}{1638}$, $\frac{1521}{1638}$.

(12) $\frac{11}{111}$, $\frac{11}{111}$, $\frac{11}{111}$, $\frac{74}{111}$.

(13) $\frac{1260}{1260}$, $\frac{840}{1260}$, $\frac{630}{1260}$, $\frac{504}{1260}$, $\frac{480}{1260}$, $\frac{360}{1260}$, $\frac{315}{1260}$, $\frac{280}{1260}$.

(14) $\frac{1260}{2520}$, $\frac{1680}{2520}$, $\frac{1890}{2520}$, $\frac{2016}{2520}$, $\frac{2100}{2520}$, $\frac{2160}{2520}$, $\frac{2205}{2520}$, $\frac{2240}{2520}$.

(15) $\frac{60}{60}$, $\frac{30}{60}$, $\frac{20}{60}$, $\frac{15}{60}$, $\frac{12}{60}$.

(16) $\frac{235980}{8495280}$, $\frac{246240}{8495280}$, $\frac{37945}{8495280}$, $\frac{27968}{8495280}$.

(17) $\frac{17}{204}$, $\frac{17}{204}$, $\frac{17}{204}$, $\frac{2}{204}$.

(18) $\frac{3600}{4000}$, $\frac{2250}{4000}$, $\frac{360}{4000}$, $\frac{225}{4000}$, $\frac{36}{4000}$.

7. In order of value the fractions will stand thus :—

(1) $\frac{2}{3}$, $\frac{1}{2}$, $\frac{2}{5}$. (2) $\frac{5}{3}$, $\frac{3}{4}$, $\frac{2}{3}$. (3) $\frac{5}{6}$, $\frac{7}{10}$, $\frac{3}{5}$.

(4) $\frac{23}{24}$, $\frac{7}{8}$, $\frac{9}{15}$. (5) $\frac{27}{18}$, $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{6}{15}$.

(6) $\frac{3}{7}$ of $2\frac{1}{3}$, $\frac{29}{36}$, $\frac{2}{9}$ of $\frac{3}{8}$. (7) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$.

(8) $\frac{8}{9}$, $\frac{7}{8}$, $\frac{6}{7}$, $\frac{5}{6}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{2}$. (9) $\frac{5}{9}$, $\frac{4}{8}$, $\frac{3}{7}$, $\frac{2}{5}$, $\frac{2}{6}$.

(10) $\frac{6}{7}$, $\frac{5}{8}$, $\frac{3}{6}$, $\frac{4}{9}$, $\frac{2}{5}$, $\frac{1}{4}$, (11) $\frac{42}{51}$, $\frac{32}{41}$, $\frac{22}{31}$, $\frac{12}{21}$.

(12) $\frac{7}{12}$, $\frac{13}{25}$, $\frac{9}{20}$, $\frac{7}{35}$, $\frac{12}{273}$. (13) $\frac{28}{24}$, $\frac{23}{25}$, $\frac{12}{16}$, $\frac{32}{56}$, $\frac{24}{128}$.

(14) $\frac{15}{21}$, $\frac{10}{24}$, $\frac{5}{18}$, $\frac{25}{105}$. (15) $\frac{222}{900}$, $\frac{22}{100}$, $\frac{2}{10}$, $\frac{90}{1000}$.

Ex. X.

1. (1) $\frac{3}{4}$. (2) $\frac{5}{6}$. (3) $\frac{7}{10}$. (4) $\frac{29}{30}$.
(5) $1\frac{3}{14}$. (6) $1\frac{16}{21}$. (7) $5\frac{1}{4}$. (8) $8\frac{17}{28}$.
(9) $19\frac{17}{30}$. (10) 12. (11) $1\frac{1}{6}$. (12) $\frac{103}{216}$.
(13) $2\frac{23}{24}$. (14) $2\frac{89}{120}$. (15) $40\frac{691}{2925}$. (16) $121\frac{63}{88}$.
(17) $5\frac{9511}{32032}$. (18) $\frac{871}{1170}$.

2. (1) $1\frac{17}{60}$. (2) $\frac{1237}{1260}$. (3) $\frac{241}{420}$. (4) $\frac{3589}{3960}$.
(5) $1\frac{1}{4}$. (6) 1. (7) $1\frac{121}{280}$. (8) $52\frac{179}{2520}$.
(9) $\frac{73}{126}$. (10) $5\frac{11}{120}$ (11) $31\frac{23}{70}$. (12) $108\frac{47}{264}$.
(13) $29\frac{329}{440}$. (14) $29\frac{297}{1400}$. (15) $84\frac{683}{1224}$.

Ex. XI.

1. (1) $\frac{1}{6}$. (2) $\frac{1}{15}$. (3) $\frac{1}{10}$. (4) $\frac{1}{12}$.
(5) $\frac{1}{20}$. (6) $\frac{1}{20}$. (7) $\frac{3}{20}$. (8) $\frac{1}{30}$.

(9) $\frac{1}{48}$. (10) $4\frac{1}{2}$. (11) $3\frac{1}{12}$. (12) $4\frac{53}{60}$.
(13) $20\frac{19}{45}$. (14) $24\frac{2}{3}$. (15) $17\frac{1}{33}$. (16) $\frac{23}{116}$.
(17) $8\frac{2}{111}$. (18) $1\frac{1}{30}$. (19) $1\frac{1}{3}$. (20) $11\frac{2}{3}$.

2. (1) $\frac{31}{60}$. (2) $\frac{104}{315}$. (3) $1\frac{13}{150}$. (4) $\frac{49}{120}$.
(5) $\frac{79}{462}$. (6) $\frac{337}{3300}$.

3. $\frac{1}{2}$. 4. $\frac{8}{15}$. 5. $\frac{1}{30}$. 6. $\frac{3}{8}$.

Ex. XII.

1. (1) $\frac{1}{3}$. (2) $\frac{1}{2}$. (3) 1. (4) $\frac{1}{4}$.
(5) $\frac{5}{14}$. (6) $\frac{5}{6}$. (7) $\frac{3}{14}$. (8) $\frac{5}{16}$.
(9) $\frac{9}{35}$. (10) $1\frac{1}{3}$. (11) $\frac{9}{25}$. (12) $\frac{1}{16}$.

2. (1) $\frac{1}{30}$. (2) $\frac{1}{9}$. (3) $\frac{4}{539}$. (4) $\frac{3}{14}$.
(5) $\frac{49}{64}$. (6) $\frac{1}{50}$

3. (1) $1\frac{7}{15}$. (2) $1\frac{113}{210}$. (3) $3\frac{91}{132}$.

Ex. XIII.

1. (1) 2. (2) 3. (3) 4. (4) $\frac{3}{8}$.
(5) 9. (6) $\frac{2}{3}$. (7) 9. (8) $3\frac{3}{7}$.
(9) $\frac{15}{64}$. (10) $1\frac{3}{7}$. (11) 72. (12) $1\frac{83}{243}$.

2. $2\frac{3}{7}$. 3. 20. 4. 9.

Ex. XIV.

I.

3. $\frac{89}{91}$. 4. $\frac{3}{125}$. 5. 1. 6. The former.

II.

1. $\frac{27}{35}$, $\frac{6}{35}$. 2. 30 feet. 3. 45 years, 20 years, and 15 years.
4. 30, 6, 5. 5. $\frac{113}{120}$. 6. $\frac{7}{12}$.

III.

1. $\frac{1}{12}$. 2. $\frac{5}{96}$. 3. $\frac{17}{30}$. 4. 1800, 400. 5. $\frac{3}{20}$. 6. 36, 8.

IV.

2. $\frac{16}{81}$. 3. 5. 4. 14 years and 10 years. 5. $\frac{7}{20}$. 6. 240.

3. 20. 4. $\frac{4}{9}$. 5. $\frac{7}{8}$. 6. $\frac{\text{[illegible]}}{5}$.

VI.

1. (1) $\frac{5}{81}$. (2) $1\frac{1}{6}$. 2. $\frac{1}{10}$, $\frac{3}{10}$. 3. 36, 18, 12. 4. $2 \times 3\frac{3}{4}$.
5. $\frac{8}{9}$, $\frac{4}{9}$ 6. 80 parts.

Ex. XV.

1. (1) ·3 ; ·7 ; ·05 ; ·66 ; 5·05 ; 660·69 ; 100·01.
   (2) ·001 ; 99·009 ; 123·045.
   (3) 1000000·000001 ; 5000050·002053.

2. (1) Two hundredths ; one and three hundredths ; twenty-one and twelve hundredths ; one and one ten-thousandth ; twenty and two ten thousandths.

(2) One hundred and twenty-three and four hundred and fifty-six thousandths ; seven thousand eight hundred and ninety-one and one thonsand one hundred and twelve hundred-thousandths ; thirteen and one thousand five hundred and seventeen ten-thousandths.

(3) One millionth ; fifty hundred-millionths ; five hundred hundred thousandths.

3. (1) $\frac{1}{10}$; $\frac{13}{100}$; $12\frac{34}{100}$; $567\frac{8}{10}$; $\frac{1}{100}$.
   (2) $\frac{1}{100000}$; $\frac{2}{1000}$; $\frac{3}{100}$; $100\frac{2}{1000}$.
   (3) $35\frac{970}{1000}$; $\frac{300}{100000}$; $12\frac{321}{1000}$.

4. (1) $\frac{1}{4}$; $2\frac{1}{2}$; $\frac{1}{400}$; $\frac{1}{400}$.
   (2) $56\frac{16}{25}$; $72\frac{7}{2500}$; $1\frac{1}{500}$.
   (3) $\frac{16}{125}$; $17\frac{7}{25}$; $\frac{3}{625}$; $61\frac{16}{25}$.

5. (1) ·1, ·5, 2·3, 5·3, 9·2.
   (2) ·0011, 32·109, 68·14, 57·2.
   (3) 24·69, 1234·56789, 1·00200.

6. (1) 3, 30, 300. (2) .03, ·3, 3.
   (3) 2000·020, 200002. (4) 1560, 15600.
   (5) 202, 2020.

7. (1) ·003, ·000003. (2) ·03156, ·003156.
   (3) ·35671, 356·71. (4) 987·65005, 98·765005.
   (5) ·031415, ·00031415.

Ex. XVI.

1. (1) 1371·65295. (2) 1234·62345.
(3) 12·061623. (4) 13703·70569.
(5) 1583·00955. (6) 80641·003579.

2. (1) 200·387053. (2) ·616077. (3) 1050000·170001.

3. (1) 446·806. (2) 755·5863. (3) 1017·191.
(4) ·00535823. (5) 2596·217. (6) 388·85.

Ex. XVII.

1. (1) 44·37. (2) 44·44. (3) 1·97. (4). 1·28
(5) 18·8944. (6) 643·9904.

2. (1) ·9 (2) 2·97. (3) ·999999. (4) 999999·999999.
(5) 6·3 (6) ·81.

3. (1) ·014704380. (2) ·29001. (3) 1·3653. (4) 46·559.
(5) 96·656. (6) 5594·5056.

Ex. XVIII.

1. (1) 108·78. (2) 330·5565. (3) 5·29.
(4) ·3136. (5) 1074·176103. (6) 35·940476.

2. (1) 1. (2) ·056. (3) ·1.
(4) 44·89. (5) 5. (6) ·18.

3. (1) ·004096. (2) ·196928646. (3) 45722·668222.
(4) ·00000000003752. (5) ·66. (6) 5·14411722.

Ex. XIX.

1. (1) ·24. (2) 170. (3) 2·69. (4) 6·712.
(5) 30300. (6) ·003. (7) 1133·67. (8) 13490700.
(9) 1179000. (10) 2532. (11) 12132. (12) 420.

2. (1) 18·6423. (2) 659204·3478. (3) 5778·2962.
(4) ·0008. (5) 3.6366. (6) ·0002. (7) ·9348.
(8) ·2777. (9) 1115·7190. (10) 9·5890.

3. (1) 1000. (2) 10. (3) 100. (4) ·011. (5) 100000. (6) 19.

4. 252. 5. 560. 6. ·08125.

Ex. XX.

1. (1) ·5, ·25, ·125, ·0625, ·03125. (2) ·5, ·2, ·75, ·2, ·8.
(3) ·2, ·04, ·008, ·0016. (4) ·75, ·8, ·625, ·8, ·04.
(5) ·44, ·9765625, ·8, ·4. (6) ·59375, ·32, ·25, ·375.

2. (1) ·24324, ·30769, ·46666, ·58333, ·90909.
(2) ·33333, ·14285, ·11111, ·09090, ·07692.
(3) 13333, ·17647, ·21052, ·23809, ·26086.
(4) ·91666, ·92307, ·92857, ·93333.
(5) ·95454, ·95652, ·96333, ·96153
(6) 7·42857, 8·44444, ·05785, ·04733.

3. (1) $\cdot\dot{4}2857\dot{1}$, $\cdot\dot{5}$, $\cdot\dot{8}\dot{1}$, $\cdot\dot{8}46153\dot{3}$, $\cdot9\dot{2}8571\dot{4}$, $\cdot9\dot{3}$.
(2) $\cdot\dot{3}8461\dot{5}$, ·952380, $\cdot8\dot{3}$, $\cdot\dot{5}7142\dot{8}$.
(3) $\cdot\dot{4}2857\dot{1}$, $\cdot\dot{4}2857\dot{1}$, $\cdot\dot{6}$, $\cdot\dot{4}\dot{5}$. (4) $\cdot\dot{7}$, $\cdot21\dot{4}285\dot{7}$, $\cdot02\dot{2}8571\dot{4}$, $\cdot\dot{2}$.
(5) $\cdot\dot{3}$, $\cdot\dot{4}2857\dot{1}$, $\cdot\dot{5}3846\dot{1}$, $\cdot\dot{8}\dot{1}$. (6) $\cdot\dot{7}$, $\cdot\dot{6}9$, $\cdot08\dot{3}$, $\cdot01\dot{6}$.

4 (1) $\frac{7}{9}$, $\frac{89}{99}$, $\frac{9}{11}$, $\frac{31}{33}$, $\frac{41}{333}$. (2) $5\frac{71}{90}$, $5\frac{62}{99}$, $2\frac{17}{45}$, $9\frac{8}{45}$.
(3) $1\frac{11}{30}$, $5\frac{11}{15}$, $6\frac{28}{99}$, $\frac{34}{45}$. (4) $\frac{13}{18}$, $\frac{5}{6}$, $\frac{73}{90}$, $8\frac{5}{9}$, $\frac{61}{90}$.
(5) $\frac{8}{90}$, $\frac{49}{9000}$, $\frac{679}{999}$, $\frac{41}{111}$. (6) $\frac{1}{333}$, $\frac{1}{300}$, $\frac{7}{90}$, $\frac{65}{999}$.

Ex. XXI.

1. (1) ·08255, ·12346, ·98765. (2) ·13579, ·24681, ·51153.
(3) ·86429, ·13035, ·72337. (4) ·27333, ·02737, ·09232.

2. (1) 70·37035. (2) ·40972. (3) 7·45606.
(4) 63·6759. (5) 84·20465. (6) 8·51.
(7) 2·46913. (8) 335·14620. (9) 204·31234
(10) 1·3; ·3037; ·03198. (11) ·1237; ·4193; 12·2.
(12) ·8; 3·69135; 67·02469.
(13) 1·24999. (14) 1·45000.
(15) ·05, ·2, ·0185, 45.

Ex. XXII.

1.

[illegible]

II.

1. 8·3751. 2. 4798·902$\dot{6}$. 3. 1000·8999.
4. ·3375, 0. 5. ·615, ·115. 6. 2 × 2·357.

III.

1. ·18305, 18304 81695. 2. 10 years, 5 years. 3. 14, 8.
4. 100 Rs , 150 Rs., and 540 Rs. 5. ·2.
6. 1000 Rs., 350 Rs., 150 Rs.

IV.

1. ·25, 2·5, 2500. 2. 70000. 3. ·0464.
4. The 3rd is the greatest, and the 1st the least. 5. 1·7598. 6. ·8.

V.

2. ·$\dot{1}$17647058823529$\dot{4}$. 3. $\frac{15}{768}=\frac{5}{256}=\frac{5}{2^8}$

4. ·12$\dot{5}$6$\dot{7}$, ·$\dot{4}$0$\dot{5}$. 6. $\frac{14}{37}$, $\frac{29}{55}$, ·$\dot{2}$8571$\dot{4}$, ·5625.

VI.

1. ·7853. 2. 6·08$\dot{3}$. 3. 1·05750. 4. 4, 12, 27.
5. ·7853. 6. 1·8289.

Ex. XXIII.

1. (1) 2464p. ; 5 Rs. 3a. 4p. (2) 4489p. ; 15 Rs. 10a.
(3) 30535p. ; 31 Rs. 4a. (4) 3759a. ; 12 Rs. 4a.
(5) 5067*d.* ; £2. 1*s.* 8*d.* (6) 30882*q.* ; £0. 8*s.* 4*d.*
(7) 27000*d.* ; 44 half-guineas, 7*s.* 5*d.* (8) £66. 2*s.* 6*d.* ; 209 crowns.
(9) 6792 grs. ; 2oz. 11dwts. 10grs.
(10) 3154dwts. ; 21 lbs. 5oz. 2dwts. 6grs.
(11) 8372 lbs. ; 62 lbs. 8oz.
(12) 90272000 grs. ; 4 tons, 9 cwt. 1 qr. 4 lbs.
(13) 2412 rattis. ; 10 tolas 40 rattis.
(14) 81420 kanchas; 12½ seers. (15) 320000 tolas ; 3 mds. 5 seers.
(16) 3371½ yds. ; 1 mile 240 yds. (17) 859 in. ; 14 yds. 1 ft. 9 in.
(18) 7350¾ sq. yds. ; 0 ac. 2 ro. 19 p. 5¼ sq. yds.
(19) 25947 cub. in. ; 18 cub. yds. 14 cub. ft.
(20) 8200 cubits ; 1 kros 95 rasis 78 cubits.
(21) 41600 cubits ; 10 kroses. (22) 575 kths. ; 26 bghs. 3 kths.
(23) 796 kths. ; 5 bghs. 3 kths. 2 chts.

(24) 1230 grams ; 987·654 grams.

(25) 456000 centimetres ; 78·91 dekam.

2. 24000*d.* ; £6. 13*s.* 4*d.* 3. 409600 acres ; 7 bghs. 4 kths.

4. 525960 min.; 420 dandas.

Ex. XXIV.

1. (1) £23. 7*s.* $7\frac{1}{2}$*d.* (2) £89. 13*s.* $3\frac{1}{4}$*d.* (3) £152. 15*s.* 11*d.*

(4) 136 Rs. 6a. 6p. (5) 179 Rs. 6a. 10p. (6) 289 Rs. 2a. 8p.

(7) 149 tons 8 cwt. 14 lbs. (8) 48 tons 11 cwt. 3 qrs. 26 lbs.

(9) 137 mds. 20 seers 11 chts. (10) 210 mds. 25 seers 8 chts.

(11) 32 yds. 1 ft. 7 in. (12) 207 miles 6 fur. 33 po.

(13) £69. 16*s.* $7\frac{1}{2}$*d.* (14) £103. 12*s.* $4\frac{1}{4}$*d.*

(15) £209. 7*s.* $\frac{3}{4}$*d.* (16) 260Rs. 7a. 2p.

(17) 109Rs. 14a. 9p. (18) 141Rs. 14a 2p.

2. 39 bghs. 13 kths. 7 chts. 3. 77 sq yds. 6 sq. ft. 30 sq in.

4. 27 lbs. 5 oz 4 dwts. 5. 208Rs. 12 a. 8 p.

6. 108 miles 2 fur. 31 po.

Ex. XXV.

1. (1) £10. 18*s.* $9\frac{3}{4}$*d.* (2) £46. 13*s* 10*d.* (3) £48. 12*s.* $3\frac{1}{2}$*d.*

(4) £265. 11*s.* 1*d.* (5) 3Rs. 0a. 11p. (6) 17Rs. 14a. 9p.

(7) 39Rs. 9a. 9p. (8) 22Rs. 0a. 6p. (9) 24lbs.. 8oz. 18dwts.

(10) 17cwt. 2qrs. 22lbs. 15oz. 1dr. (11) 13mds. 30seers 12 chts.

(12) 11mds. 19seers 3chts. (13) 68miles 3fur. 34po.

(14) 2kroses 95 rasis. (15) 16 sq. yds. 7sq. ft. 108sq. in.

(16) 8cub. ft. 1721 cub. in. (17) 8bghs. 17kths. 14cht.

(18) 2 kilog. 7 hectog. 8 dekag. 1gram.

2. (1) 7 weeks 4 days 19 hrs. (2) 5 dandas 44 pals.

(3) 15hrs 54′ 57″ (4) 12° 55′ 55″.

(5) 5a. 7gan. 2cowr. 2kr. (6) 2a. 13gan. 1cowr. 1kr.

Ex. XXVI.

1. (1) £31 1*s.* ; £46, 11*s.* 6*d.* ; £77. 12*s.* 6*d.*

(2) £66. 6*s.* 6*d.* ; £82. 18*s.* $1\frac{1}{2}$*d.* ; £99. 9*s.* 9*d.*

(3) £125 4*s.* $8\frac{1}{4}$*d.* ; £143. 2*s.* 6*d.* ; £161. 0*s.* $3\frac{3}{4}$*d.*

(4) £286. 6*s.* 8*d.* ; £343. 12*s.* ; £429. 10*s.*

(5) 166Rs 6a. ; 249Rs. 9a. ; 332Rs. 12a.

(6) 129Rs. 12a. 3p.; 155Rs. 11a. 6p.; 207Rs. 10a.
(7) 130Rs. 2a.; 195Rs. 3a ; 390Rs. 6a.
(8) 170Rs. 10a. 9p.; 341Rs. 5a. 6p.; 512Rs. 0a. 3p.
(9) 40lbs. 3oz. 3drs. 2scr.; 50lbs. 4oz, 2drs. 1scr. 15grs; 100lbs. 8oz. 5drs. 10grs.
(10) 78cwt. 2qurs.; 157cwt.; 235cwt. 2qrs.
(11) 31mds. 24 seers 2powas; 42mds. 6seers; 47mds. 16seers 3powas
(12) 260mds. 39seers 1cht.; 347mds. 38seers 12chts; 556mds. 23seers.
(13) 42wks. 11hrs. 30′; 56wks, 15hrs. 20′; 82wks. 5days 5hrs. 20′.
(14) 182days 10hrs. 45′; 188days 8hrs.; 235days 10hrs.
(15) 32kilog. 5hectog. 3dekag. 3decig 5centig. 65 kilog. 6dekag. 7decig.; 97kilog. 5hectog. 9dekag 1gram. 5centig
(16) 1hectom. 7dekam 5metres 2decim; 3hectom. 5dekam. 4decim.; 7hectom. 8decim.
2 (1) *£46. 12s. 6d* ; *£466 5s.*
(2) 186Rs. 9a.; 1865Rs. 10a.
(3) 785cwt. 2qrs. 24lbs.; 1571cwt. 1qr. 20lbs
(4) 321lbs. 3oz. 11dwts. 16grs.; 3212lbs. 11oz. 16dwts. 16grs.
(5) 154mds. 33 seers 2chts.; 1548mds. 11seers. 4chts.
(6) 305mds. 5seers; 457 mds. 27seers 8chts.

Ex. XXVII.

1. (1) £4. 15*s.* $2\frac{1}{2}$*d.*; £3. 3*s.* $5\frac{2}{3}$*d.*; £2. 7*s.* $7\frac{1}{4}$*d.*
(2) £11. 3*s.* $1\frac{1}{5}$*d.*; £9. 5*s.* 11*d.*; £7. 19*s.* $4\frac{3}{7}$*d.*
(3) £8. 9*s.* $8\frac{3}{8}$*d.*; £7 10*s.* $10\frac{1}{9}$*d.*; £6. 15*s.* $9\frac{7}{10}$*d.*
(4) £3. 2*s.* $1\frac{5}{9}$*d.*; £3. 2*s.* $1\frac{15}{73}$*d.*; £3. 0*s.* $5\frac{1}{3}$*d.*
(5) 5Rs. 11a. $\frac{3}{10}$p.; 4Rs. 11a. $10\frac{1}{4}$p.; 4Rs. 1a. $\frac{3}{14}$p.
(6) 10Rs. 0a. $10\frac{2}{15}$; 9Rs. 6a. $9\frac{1}{2}$p.; 8Rs. 6a. $\frac{4}{9}$p.
(7) 11Rs. 14a. $\frac{10}{19}$p.; 7Rs. 4a. $5\frac{23}{31}$p.
(8) 80Rs. 1a $\frac{1}{4}$p.; 35Rs. 9a. $4\frac{1}{8}$p.
(9) 14 gals. $\frac{3}{4}$ qts.; 4 gals. $\frac{3}{14}$ qts.
(10) 9 qrs. $1\frac{1}{4}$ pks.; 4 qrs. 4 bus. $\frac{5}{8}$ pks.
(11) 1 cwt. 3 qrs. $23\frac{1}{8}$ lbs.; 3 qrs. $19\frac{7}{16}$ lbs.

(12) 2 cwt. 2 qrs. $4\frac{4}{5}$ lbs.; 1 cwt. 3 qrs. $17\frac{3}{5}$ lbs.

2. (1) $12\frac{19}{102}$. (2) $8\frac{252}{253}$. (3) $15\frac{111}{667}$. (4) $4\frac{9}{62}$.
   (5) 19166400. (6) $4\frac{18}{47}$. (7) $4\frac{26}{51}$ (8) 3.
   (9) $7\frac{9}{35}$. (10) $5\frac{5}{39}$. (11) $4\frac{5}{42}$ (12) $198\frac{14}{17}$.

Ex. XXVIII.

I.

2. $87\frac{3}{11}$ grs.; $2\frac{1}{16}$s. 3. 400 drs.; $877\frac{5}{7}$ drs.
4. 3960q.; $17\frac{3}{16}$ Rs. 5. 4752000000. 6. 126.

II.

1. 187500 mds.; 18750000 lbs. Troy; 11719 carts, the last cart carrying only 12 mds.
2. 39062 Rs. 8a.; 1002604 Rs. 2a. 8p.
3. 30000000 mds.; 150000000 Rs. 5. 113565760 bghs.
6. 8766 mds.

III.

1. £4 6*s*. 4*d*. 2. 5600 pice.
3. 21 years 45 days; 19th May, 1850. 4. 29 Rs. 10a.
5. 27 mds. 18 seers; 137 Rs. 4a. 6. 8180 days.

IV.

2. 44 Rs. 14a. 6p.; 479 miles.
3. 25 rupees, 50 half rupees, 75 four anna pieces, 100 two anna pieces.
4. 14 Rs. 1a. 3p. 5. 5 mds. 33 seers. 6. 25168000000 bghs.

V.

1. 275 Rs. 8a. 2. 4 Rs. 13a. 3. 1950 Rs.
4. 56940. 5. 12. 6. 5.

VI.

1. 45. 2. 15. 3. 3 days. 4. 8′ 20″.
5 A gets 400 Rs., B, 600 Rs., and C, 800 Rs.
6. 196 Rs. 10a.; 3 Rs. 1a. $1\frac{7}{8}$p.

Ex. XXIX.

1. (1) £1. 2*s*. 6*d*.; £3. 3*s*. 8*d*.
   (2) £12. 13*s*. 6*d*.; £2. 10*s*. 2*d*.
   (3) £3. 19*s*.; £33. 15*s*.
   (4) £7. 17*s*. 11*d*.; £26. 12*s*.

(5) 2 Rs. 14a. $2\frac{2}{3}$p.; 5 Rs. 0a. 8p.
(6) 13a. 1p.; 1R. 5a. 1p.
(7) 13 cwt. 2 qrs. 14 lbs. 14 oz.; 1 lb. 7 oz. 13 dwts.
(8) 7 mds. 13 seers 4 chts.; 7 mds. 19 seers 4 chts.
(9) 6 fur. 12 po.; 1 yd. 2 ft. 9 in.
(10) 2 days 2 hrs. 21′; 2 hrs. 39′.

2. (1) $\frac{3}{4}$; ·8. (2) $\frac{1}{42}$; ·25. (3) $\frac{2}{3}$; ·5.
(4) $\frac{1}{18}$; ·1. (5) $\frac{1}{48}$; ·03125. (6) $\frac{3}{8}$; ·125.
(7) $\frac{5}{16}$; ·25. (8) $\frac{2}{5}$; 00625.

3. $\frac{5}{8}$; ·3. 4. $\frac{1}{40}$; ·125. 5. $\frac{1}{4}$; $\frac{1}{16}$.

6. In order of value, the quantities will stand thus :—
(1) $\frac{1}{15}$ of 1R. 8a., $\frac{1}{12}$ of 1R., $\frac{2}{3}$ of 2a.
(2) $\frac{1}{3}$ of £1., $\frac{3}{8}$ of 2 crowns, $\frac{5}{6}$ of 1 florin.
(3) $\frac{3}{8}$ of 15 seers, $\frac{7}{8}$ of 1 pusury, $\frac{1}{15}$ of 1 md.
(4) $\frac{5}{9}$ of 3 ft., $\frac{3}{8}$ of 4 ft., $\frac{1}{15}$ of 2 yds.

Ex. XXX.

(1) £1. 2*s*. 4*d*. (2) 16*s* $6\frac{2}{5}$*d*. (3) 4 Rs. 14a. $10\frac{2}{3}$p.
(4) 18*s*. 6*d*. (5) 2*s*. $5\frac{11}{30}$*d*. (6) 3 Rs. 10a. $2\frac{2}{5}$p.
(7) 2 mds. 14 seers 5 chts. (8) 2 ft. $11\frac{1}{4}$ in.
(9) 3 days 23 hrs. (10) 1 qr. 11 lbs. 10 oz.

Ex. XXXI.

(1) 0. (2) 3*s*. 9*d*. (3) £3. 13*s*. (4) 3 Rs. 1a. 8p.
(5) 2a. 4p. (6) 1 R. (7) 2 qrs. 11 lbs. (8) 2 in.
(9) 16 seers. (10) 2 hrs. 28′ 40.″

Ex. XXXII.

(1) 17*s*. $\frac{2}{3}$*d*. (2) £6. 19*s*. 4*d*. (3) £7. 1*s*. $10\frac{3}{4}$*d*.
(4) 1 R. 4a. $\frac{3}{4}$p. (5) 17 Rs. 6a. $\frac{3}{5}$p. (6) 19 Rs. 7a. $9\frac{1}{4}$p.
(7) 1 qr. 9 lbs. 8 oz, (8) 15 seers $2\frac{1}{5}$ chts. (9) $8\frac{3}{20}$*d*.
(10) 6a $2\frac{1}{4}$p. (11) 1 ft. $2\frac{4}{9}$ in. (12) 17*s*.

Ex. XXXIII.

(1) £13. 14*s*. $10\frac{3}{4}$*d*. (2) 75. (3) 12a. (4) 4.

(5) 16 Rs. 9a. (6) 5 ft. (7) 6 ft. 8 in. (8) $\frac{22}{35}$.

(9) 10. (10) $\frac{1}{3}$. (11) 1 bus. $3\frac{1}{3}$ pks. (12) 2.

Ex. XXXIV.

1. 560 Rs. 2 672 Rs. 3. 1066 Rs. 10a. 8p.

4. $94\frac{29}{64}$ S. Rs. 5. £25. 1*s*. 3*d*. 6. 162 Rs. 12a.

7. $8\frac{19}{28}$ Factory mds.; 5 cwt. 3 qrs. 4lbs.

8. 105 lbs. Troy; $86\frac{2}{5}$ lbs. Avoir. 9. 12 lbs. Avoir.

10. $20\frac{5}{96}$ lbs. Troy. 11. 40656 bghs. 12. $145\frac{75}{121}$ acres.

13. $38\frac{91}{160}$ bghs. 14. $13\frac{1}{3}$ dandas; $6\frac{1}{10}$ hrs. 15. $\frac{64}{75}$ hr.; $22\frac{11}{12}$ dandas.

Ex. XXXV.

I.

1. $2\frac{113}{296}$s. 2. 9600Rs. 3. $11\frac{11}{24}$srs.; $28\frac{1}{33}$lbs. Troy.

4. 560Rs.; £51. 6*s*. 8*d*. 5. 2100Rs.; 1470Rs.

6. $\frac{1}{2}$; 3262Rs. 8a.

II.

1. 4840sq. yds.; 1600sq. yd.; 1936bghs.; $198\frac{49}{121}$ ac.

2. $2\frac{1}{2}$ miles an hour. 3. $\frac{3}{40}$; $616\frac{7}{8}$bghs.

4. $\frac{1}{640}$; $\frac{40}{121}$. 5. ·0125; $\frac{1}{1760}$.

6. 330ft.; $\frac{4}{11}$.

III.

1. 228Rs. 9a. 6p. 2. 65625Rs.

3. *A* gets 800Rs., *B*, 1200Rs., and *C*, 1500Rs.

4. *A* gets 400Rs., *B*, 300Rs, and *C*, 90Rs.

5. $\frac{1}{100}$; $\frac{36}{49}$.

6. $33\frac{1}{3}$ mds. of the first kind, and $66\frac{2}{3}$mds. of the second kind.

IV.

1. 64Rs. 0a, $6\frac{1}{2}$p. 2. 6 days. 3. $395\frac{215}{2035}$.

4. 7bghs. $17\frac{1}{2}$kths. 5. $266\frac{2}{3}$yds. 6. $1\frac{107}{433}$.

V.

1. At 5 o'clock. 2. $3\frac{2}{9}$hrs. 3. 365·242218 days; 365·25days; ·007782 of a day. $128\frac{1952}{3891}$ years.
4. 52 Sundays; 4 Saturdays.
5. $\frac{11}{25}$; 26400 jojans. 6. $11\frac{344985639}{485552800}$ seconds.

VI.

1. 14400grs.; 15432·3487grs. 2. $\cdot 621382\dot{5}\dot{7}$.
3. 30miles from A; 12hrs.; 2 hrs. 12'.
4. 2days. 5. 15gals. 6. In 1883.

Ex. XXXVI.

1. (1) 3 sq ft. 54 sq in. (2) 9 sq. ft 90 sq. in.
   (3) 14 sq. ft. 126 sq in. (4) 26 sq. ft. 120 sq. in.
   (5) 34 sq. ft. 120 sq. in. (6) 10 sq. ft. 32 sq. in.
2. (1) 2 cub. ft. 918 cub. in. (2) 45 cub. ft. 972 cub. in.
   (3) 11 cub. ft. 576 cub. in. (4) 33 cub. ft. 432 cub. in.
3. 277 sq. ft. 72 sq. in. · 74 ft. 4. 265 sq. ft. 90 sq. in.; 26 ft. $6\frac{3}{4}$ in.
5. 125 cub. in.; 420 cub. ft.
6. (1) 3 bghs. 12 kths. (2) 7 bghs. 17 kths. 10 dhools.
   (3) 13 bghs. 12 kths. (4) 25 bghs. 3 kths. 10 dhools.
   (5) 17 bghs. 17 kths. (6) 42 bghs. 10 kths.

Ex. XXXVII.

1. 3 sq ft. 5'. 3." 2. 15 sq ft. 6'. 8." 3. 21 sq. ft. 1.'
4. 34 sq ft 6'. 5. 50 sq. ft. 5'. 8." 6. 50 sq. ft. 6'. 8".
7. 60 sq ft. 6' 3". 8. 58 sq. ft. 1'. 6." 9. 33 sq. ft. 3'. 9".
10. 58 sq. ft. 8' 2."

Ex. XXXVIII.

1. 33 Rs. 2. 23 bghs. 17 kths. 3. £23. 1s. $1\frac{1}{3}d$.
4. 27 Rs. 9 a. 5. 4 bghs. 6. 1410 Rs.
7. 240 Rs 8. 44 yds. 9. 51840 Rs.
10. 27 ft. 11. 2479 Rs. 8 a. 12. 45".
13. $140\frac{70087}{128637}$. 14. 12 ft. 6 in. 15. 96.
16. 1152 bricks; 10 Rs. 15 a. $1\frac{31}{125}$p. 17. 63.

18. 42 cub. ft 1512 cub. in.
19. 32 in the 1st class, 62 in the 2nd, and 192 in the 3rd.
20. 3536 Rs.

Ex. XXXIX.

1. 106Rs. 4.a. ; 212Rs. 2. £ 108 ; £ 42. 12 *s.* 6 *d.*
3. £ 84 ; £ 225. 4. 71 Rs. 4 a ; 6 Rs. 12 a. 6 p.
5. 92 Rs. 8 a. ; 141 Rs. 4 a. 6. 275 Rs. ; 230 Rs. 10 a.
7. £ 2376 ; £ 420. 7 *s.* 6 *d.* 8. £ 2858. 8*s* ; £ 2108 8*s.*
9. £ 15845. 16 *s.* 8 *d.* ; £ 5087. 10*s.*
10. 2312 Rs. 8 a. ; 5156 Rs. 4 a.
11. 6746 Rs. 4 a. ; 5361 Rs. 15 a 8 p.
12. 6114 Rs. 7a. ; 23437 Rs. 8a. 13. £218. 5*s.*
14. £43. 15*s.* $10\frac{1}{2}$*d.* 15. £184. 9*s.* $3\frac{3}{7}$*d.*
16. 103 Rs. 5a. 9p. 17. 1107 Rs. 15a. $3\frac{9}{20}$p.
18. 408 Rs. 8a. 3p. 19. £18. 8*s.* 20. £21. 5*s.*
21. 11200 Rs. 8a. 22. 13216 Rs. 8a. 23. 39500 Rs.
24. 180 Rs 0a. 3p.

Ex. XL.

1. (1) 6. (2) $18\frac{2}{3}$. (3) £18. (4) 6*s.* (5) $1\frac{1}{4}$ Rs.
(6) 1, (7) $6\frac{1}{4}$ bghs. (8) 4 yds. 6 in.
2. (1) 45. (2) 14·4. (3) 3. (4) 12. (5) $1\frac{1}{4}$*s.*
(6) $\frac{1}{16}$a.
3. 30 ft. and 15 ft. 4. 20 times. 5. 18 Rs. 12a. 6. 16mds.
7. 278 Rs. 2a. 8. 20 cwt. 9. The prices are as 5 : 3 ; 6*s.* per yard. 10. 3 Rs. 12a. 11. 1396 Rs. 5a. $9\frac{9}{11}$p.
12. £600. 13. 38400 Rs. 14 7a. 15. 5500 Rs.
16. 110000 Rs. 17. 6 Rs. 18. 6 Rs.
19. 19 Rs. 11a. 20. 9 : 25. 21. 50·257...sq. ft.
22. 18 8496 ft. 23. $840\frac{40}{119}$ yds. 24. 24·5 sq ft.
25. 225000 Rs. 26. £12800.
27. 1 R. for paddy land, and 5 Rs. per mulberry land.
28. 1 R for arable land, and 2 Rs. 8a. for homestead land.
29. 773 Rs. 8a. 30. $79\frac{5}{7}$Rs. 31. *A* gets 600 Rs. and *B* gets 1000 Rs.
32. The shares of *A*, *B*, and *C* are 300 Rs. 600 Rs., and 1500 Rs.
33. $\frac{4}{11}$ of an inch ; 110 yds.

34. 1 ft. 6·75 in. 35. ·01590. 36. 6 bghs. 37. 150 poles.
38. 4 bghs. $16\frac{1}{2}$ kths. 39. 3 bghs. $7\frac{1}{2}$kths.
40. 60ft. 41. 564 miles. 42. 15 miles an hour.
43. 15 miles an hour.
44. Velocity of the Earth : velocity of hight :: 1309 : 13696875.
45. 80 ft. 46. 45 men. 47. 5 days. 48. 60 men.
49. 14 days. 50. 12 men. 51. $35\frac{305}{341}$ minutes past 12 o'clock P. M.; $24\frac{8}{17}$ minutes past 1 o'clock P. M.
52. 14′ 35″; exactly after 12 days. 53. $40\frac{1}{2}$ yds. 54. 6 days.
55. 54 men. 56. 10 mds. 57. 456 mds. 10 seers; 25mds. 20 seers. 58. 866 Rs. 14a.
59. 383 mds. 10 seers. 60. 100. 61. 450 Rs.
62. 12 oz. 63. 20. 64. $1'\frac{5}{22}''$; 490 yds.
65. 5 months. 66. £10000. 67. $38\frac{2}{11}'$ past 1 o'clock.
68. $5\frac{5}{11}'$ past 1 o'clock. 69. 18·13s. 70. 39 days.

Ex. XLI.

1. (1) 3, 6, 9. (2) 6, 9, 12. (3) 9, 12, 15. (4) 8, 24, 40, 56.
(5) 100, 200, 300, 400. (6) 165, 195, 225.
2. *A* gets 300 Rs., *B*, 900 Rs., and *C*, 1500 Rs.
3. *A* gets £1000, *B*, £800, and *C*, £600.

Ex. XLII.

1. $7\frac{1}{2}$. 2. $6\frac{3}{4}$; 2 Rs. 4a. 3. 3 Rs. 2a. 4. 8000 Rs.
5. £4000. 6. $1\frac{9}{16}$. 7. 26. 8. 2050 Rs.

Ex. XLIII.

1. 600 Rs. 750 Rs. 2. *A* gets £720, *B*, £1080, and *C*, £1200.
3. *A* should have 600 Rs. and *B*, 1800 Rs.
4. *A* should have 288 Rs., *B*, 288 Rs., and *C*, 324 Rs.
5. *A* should have 315 Rs., *B*, 280 Rs., and *C*, 315 Rs.
6. *A* should have 9500 Rs., and *B*, 2500 Rs.

Ex. XLIV.

1. (1) $4\frac{1}{2}$ Rs., $79\frac{1}{2}$ Rs. (2) $14\frac{2}{5}$ Rs., $94\frac{2}{5}$ Rs. (3) $23\frac{7}{16}$ Rs. $148\frac{7}{16}$ Rs.
(4) $1928\frac{4}{15}$ Rs., $3788\frac{4}{15}$ Rs. (5) £210 2s., £1260 12s.

(6) £844$\frac{1}{4}$, £6344$\frac{1}{4}$. (7) 425$\frac{1}{4}$ Rs., 1475$\frac{1}{4}$ Rs. (8) 225 Rs., 975Rs.

2. (1) 4$\frac{4}{5}$Rs. (2) 11Rs. 15a. 3p. (3) 18$\frac{3}{4}$Rs. (4) 31$\frac{1}{5}$Rs. (5) 13Rs. 2a.
(6) 10$\frac{1}{2}$ Rs. (7) 30$\frac{138}{365}$ Rs. (8) 128$\frac{868}{1825}$ Rs. (8) £11$\frac{17}{73}$.

3. (1) 10 years. (2) 7 years. (3) 6$\frac{2}{3}$ years. (4) 6$\frac{1}{4}$ years.
(5) 16 months. (6) 4 years.

4. (1) 16$\frac{2}{3}$ per cent. per annum. (2) 13$\frac{1}{3}$ per cent. per annum.
(3) 3$\frac{1}{8}$ per cent. per annum. (4) 1$\frac{7}{8}$ per cent. per annum.
(5) 6$\frac{1}{4}$ per cent. per annum. (6) 10 per cent. per annum.

5. (1) 500 Rs. (2) £800. (3) £500. (4) 7500 Rs.
(5) 33333$\frac{1}{3}$ Rs (6) 1600 Rs.

6. (1) £89$\frac{3}{7}$. (2) £500. (3) 6756$\frac{28}{37}$ Rs. (4) 15625 Rs.

7. 25 years. 8. 20 per cent. per annum.

Ex. XLV.

1. 16·8 Rs. 96·8 Rs. 2. 15·9792 Rs., 90·9792 Rs.
3. £10. 4*s.*, £135. 4*s.* 4. £165. 10*s.*, £665. 10*s.*
5. 163·2 Rs., 2163·2 Rs. 6. 8275 Rs., 33275 Rs.

Ex. XLVI.

1. (1) 89$\frac{3}{7}$Rs. (2) £181$\frac{9}{11}$. (3) £700. (4) 750Rs.
(5) 500Rs. (6) 1525Rs.

2. (1) £10. (2) £95. (3) 189Rs. (4) 360Rs.
(5) 333$\frac{1}{3}$Rs. (6) 535$\frac{5}{7}$Rs.

3. 8$\frac{466088}{671819}$*d.* 4. 2203$\frac{187}{371}$Rs. 5. 8951$\frac{31897}{71253}$Rs.

6. The rate of interest is 17$\frac{11}{17}$ per cent. per annum, and the rate of discount 15 per cent.

Ex. XLVII.

1. 3$\frac{4}{5}$ months. 2. After 20$\frac{2}{5}$months. 3. At the end of 7 months.
4. 4$\frac{1}{3}$months. 5. 15$\frac{1}{3}$months. 6. After 13 months.

Ex. XLVIII.

1. (1) 5000 Rs. (2) 5000 Rs. (3) 60000 Rs. (4) £4000.
(5) £9500. (6) 25000 Rs.

2. (1) 9700 Rs. (2) 12600 Rs. (3) 202 Rs. (4) £528.
(5) £1728. (6) 26500 Rs.
3. (1) 400 Rs. (2) £450 Rs. (3) £560, (4) 1800 Rs.
(5) 1900 Rs. (6) 750 Rs.
4. (1) The latter. (2) The latter. (3) The latter. (4) The former.
(5) The latter. (6) The former.
5. (1) Increase of 100 Rs. (2) No change. (3) Increase of £63.
6. (1) $4\frac{13}{47}$. (2) $3\frac{4}{7}$. (3) $4\frac{7}{17}$, (4) $5\frac{5}{14}$.
7. 108 Rs. 15a. 8. 56 Rs. 4a.

Ex. XLIX.

1. 12 Rs. 13a. 4p. 2. 1 R. 5a. 3. 4 Rs. 4. £1. 3s. 9⅓d.
5. The ingredients are mixed in equal parts. 6. 3, 1, 1.
7. 12, 4, 4. 8. 6, 2. 9. 3, 1, 1, 1.

Ex. L.

1. 3200 Rs. 2. 6000 Rs. 3. £393, 15s. 10½d. 4. 1s. 10d.
5. 1R.=⅔ of a ruble, 6. Circuitously through St. Petersburg.

Ex. LI.

1. (1) 21; 31; 99; 89; 111; 222. (2) 41; 51; 79; 69; 333.
(3) 25; 35; 45; 55; 65; 75; (4) 85; 95: 123; 234; 135.
(5) 1234; 1111; 2222; 1001. (6) 3333; 2020; 5001.
2. (1) 1, 1·4142..., 1·7320..., 2, 2·2360..., 2·4494..., 2·6457..., 2·8284..., 3, 3·1622...
(2) 3·3166..., 3·4641..., 3·6055..., 3·7416..., 9·9498..., 8·7749..., 8·1240...
(3) 1·0954..., 1·5165..., 1·8439..., 1·5297..., 1·2.
(4) ·3162..., ·0447..., ·02, 10·0049..., ·9486...
(5) ·7071..., ·5773..., ½, ·4472..., ·4082..., ·3779..., ·3535..., ⅓.
(6) ·8164..., ·5222..., $\frac{5}{12}$, $\frac{6}{19}$, $\frac{1}{3}$.
3. 7.0710...bghs 4. 155·562yds. 5. 3·1622...bghs.
6. 565·68...ft. 7. 5·6568...bghs. 8. 1·2892...miles.

Ex. LII.

1. (1) 11 ; 12 ; 13 ; 14. (2) 15 ; 16 ; 17 ; 18 ; 19.
   (3) 25 ; 111 ; 222.
2. (1) 1 ; 1·25... ; 1·44... ; 1·58... ; 1·70... ; 1·81... ; 1·91... ; 2 ; 2·08....
   (2) ·79... ; ·69... ; ·62... ; ·58... ; ·55... ; ·52... ; $\frac{1}{2}$; ·48....
   (3) ·46... ; ·58... ; ·66... ; ·1 ; ·21....
   (4) 4·97... ; 5·5 ; 6·6.
3. 14 ft. 4. 19 in.

THE END.

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